2017 Fall EE203001 Linear Algebra - Homework 6 solution

Due: 2017/12/22

1. (10%) For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}$$

Solution:

A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

2. (10%) Find the eigenvalues and unit eigenvectors of $A^T A$ and AA^T. Keep each $A\mathbf{v} = \sigma\mathbf{u}$:

$$\textbf{Fibonacci matrix } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^T$.

Solution:

$A^T A = AA^T$ has eigenvalues $\sigma_1^2 = \frac{3+\sqrt{5}}{2}$, $\sigma_2^2 = \frac{3-\sqrt{5}}{2}$.
$\sigma_1 = \frac{1+\sqrt{5}}{2} = \lambda_1(A)$, $\sigma_2 = \frac{1-\sqrt{5}}{2} = \lambda_2(A)$; $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = -\mathbf{v}_2$.

3. (10%) Write $\boldsymbol{A}$ in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \lambda_3 x_3 x_3^T$ of the spectral theorem $Q\Lambda Q^T$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \text{ (keep } \|x_1\| = \|x_2\| = \|x_3\| = 1)$$

Solution:

$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \lambda^3 - 0\lambda^2 + (-9)\lambda - 0 = 0 \rightarrow \lambda = 0, 3, -3$

when $\lambda = 0 \rightarrow \boldsymbol{x_1} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

when $\lambda = 3 \rightarrow \boldsymbol{x_2} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

when $\lambda = -3 \rightarrow \boldsymbol{x_3} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

$$bmA = 0\begin{bmatrix} 4/3 & 4/3 & -2/3 \\ 4/3 & 4/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} + 3\begin{bmatrix} 4/3 & -2/3 & 4/3 \\ -2/3 & 1/3 & -2/3 \\ 4/3 & -2/3 & 4/3 \end{bmatrix} - 3\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 4/3 & 4/3 \\ -2/3 & 4/3 & 4/3 \end{bmatrix}$$

4. (10%) Without multiplying $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, find

   (a) (2%) the determinant of $\boldsymbol{A}$.
   (b) (2%) the eigenvalues of $\boldsymbol{A}$.
   (c) (2%) the eigenvectors of $\boldsymbol{A}$.
   (d) (4%) a reason why $\boldsymbol{A}$ is symmetric positive definite.

   Solution:

   (a) $det(\boldsymbol{A}) = 1 \cdot 2 = 2$

   (b) λ =1 and 2

   (c) $x_1 = (1, -1); \quad x_2 = (1, 1)$

   (d) the λ's are positive. So $\boldsymbol{A}$ is positive definite.

5. (10%) Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

   Solution:

   Find the eigenvalues and corresponding eigenvectors of A

$$det(A - \lambda I) = 0 => \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = 0 => \lambda = 0, 2, 4$$

$$\lambda = 0 => \bar{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 2 => \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 4 => \bar{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

   Find the orthogonal matrix Q that diagonalizes A

$$\bar{E}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{E}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{E}_3 = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\rightarrow Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$

6. (10%) Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - 2x_2 + x_3)^2$$

Solution:

$A = \begin{bmatrix} 4 & -8 & 4 \\ -8 & 16 & -8 \\ 4 & -8 & 4 \end{bmatrix}$

$\text{rank}(A) = 1$

$\text{pivot} = 4$

$\text{eigenvalues} = 0, 0, 24$

$\det(A) = 0$

7. (10%)For which number b and c are these matrices positive definite?

$A = \begin{bmatrix} 1 & b \\ b & 16 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 6 \\ 6 & c \end{bmatrix}$

With the pivots in D and multiplier in L, factor each A into $LDL^{\perp}$

Solution:

$1 \times 16 - b^2 > 0,\ -4 < b < 4$
$2 \times c - 6 \times 6 > 0,\ c > 18$

$$\begin{bmatrix} 1 & b \\ b & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 \\ 6 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 18 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

8. (10%) What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

$A = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix}$

Solution:

for matrix A, $f(x, y) = 4x^2 + 12xy + 14y^2 = (2x + 3y)^2 + 5y^2$

for matrix B, $f(x, y) = x^2 + 10xy + 25y^2 = (x + 5y)^2$

9. (10%) For a nearly symmetric matrix $A = \begin{bmatrix} 1 & 10^{-19} \\ 0 & 1 + 10^{-19} \end{bmatrix}$, find out how far are it's eigenvectors (in angle) from orthogonal.

Solution:

We can find the eigenvectors to be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ by observation. The angle between the vectors is $cos^{-1}(\frac{1}{\sqrt{2}}) = 45°$, which is 45° from orthogonal.

10. (10%) Suppose A is a real antisymmetric matrix that $A^T = -A$, please show:

    (a) (3%) $\mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.

    (b) (4%) The eigenvalues of A are pure imaginary.

    (c) (3%) The determinant of A is non-negative.

    Solution:

    (a) $\mathbf{x}^T(A\mathbf{x}) = (A\mathbf{x})^T\mathbf{x} = \mathbf{x}^T A^T\mathbf{x} = -\mathbf{x}^T A\mathbf{x} \Rightarrow \mathbf{x}^T A\mathbf{x} = 0$ for every real vector $\mathbf{x}$.

    (b) Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}, \mathbf{y}$ are real vectors. First we show that $A\mathbf{z} = \lambda\mathbf{z}$ leads to $\bar{\mathbf{z}}^T A\mathbf{z} = \lambda\bar{\mathbf{z}}^T\mathbf{z} = \lambda\|\mathbf{z}\|^2$, resulting in the eigenvalue λ multiplies a real number $\|\mathbf{z}\|^2$ .Since $\bar{\mathbf{z}}^T A\mathbf{z} = (\mathbf{x} - i\mathbf{y})^T A(\mathbf{x} + i\mathbf{y})$ with real part $\mathbf{x}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{y}$, which was proved to be zero in (a) $\Rightarrow$ Any eigenvalue λ should be pure imaginary.

    (c) Since all the eigenvalues are pure imaginary, $\det(A) = \lambda_1\lambda_2...\lambda_n = (ia)(-ia)(ib)(-ib)... \geq 0$ where $a, b, ..$ are non-negative real numbers.