EE2030 Linear Algebra
Final Exam, Jan. 13, 2016
Lecturer: Yi-Wen Liu

Full score = 100. The following rules apply:

- Organize your work in a reasonably neat and coherent way so you can receive partial credit.
- **Theorems proved in class** can be referred without derivation.
- Mysterious or unsupported answers will not receive full credit.

1. (20 points) **(Gran-Schmidt orthogonalization).**

   (a) Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 12 \end{pmatrix}$. find $t \in \mathbb{R}$ such that $\langle v_2 - tv_1, v_1 \rangle = 0$.

   (b) Let V be the vector space of square-integrable continuous functions defined on $[-1, 1]$. Consider the following vectors in V: $f_3(t) = |t|$, $f_1(t) = 1$, $f_2(t) = t^2$. Find the function $g(t) \in \text{span}(\{f_1(t), f_2(t)\})$ such that $||f_3(t) - g(t)||^2$ is minimized.

2. (30 points) **(Orthogonal complement, range, and null space)**. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then we have $\mathsf{L}_A : \mathbb{R}^3 \to \mathbb{R}^2$.

   (a) Describe accurately the subspace $\mathsf{N}(\mathsf{L}_A)$ using the Gaussian elimination method. What is the dimensionality of the subspace?

   (b) Make a sketch to describe the set $\mathsf{R}(\mathsf{L}_{A^t})$.

   (c) Show that, if $y \in \mathsf{R}(\mathsf{L}_{A^t})$, and $x \in \mathsf{N}(\mathsf{L}_A)$, then $\langle x, y \rangle = 0$.

   (d) Calculate A^tA and show that it is not full-rank.

   (e) Find the eigenvalues for A^tA.

   (f) Is A^tA diagonalizable? Why or why not?

3. (30 points) **(Unitary 3x3 matrices)**. Assume that $A \in \mathsf{M}_{3x3}(\mathbb{R})$, and $A^t A = I$. In the following parts you will be guided to show that such matrices can never have three distinct real eigenvalues, and one of the eigenvalues must be 1 or -1.

   (a) Argue that A must at least have a real eigenvalue. (*Hint*: The characteristic polynomial is third-order).

   (b) Show that, if $\lambda \in \mathbb{R}$ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue for A^t. (*Hint*: $A^t = A^{-1}$).

   (c) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue for A and x is an eigenvector associated with λ. Let $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation represented by $A^* - \bar{\lambda} I$. Prove that $\mathsf{R}(\mathsf{T}) \subseteq \{x\}^{\perp}$.

   (d) Continuing from (c), if $\lambda \in \mathbb{R}$, argue that λ is also an eigenvalue for A^t.

   (e) Show that, if $\exists$ nonzero vector $x \in \mathbb{C}^3$ such that $Ax = \lambda x$, where $\lambda \in \mathbb{C}$, then $|\lambda| = 1$. (*Hint*: Unitary transformations preserves the norm).

   (f) Finally, argue that A can not have three distinct real eigenvalues, and one of the eigenvalues must be 1 or -1.

4. (20 points) **(Discriminant of a cylindrical curve)**. Let $f(x, y) = 3x^2 + 4xy + 6y^2$ be a bivariate function. Consider the curve $f(x, y) = 1$. The curve is actually an ellipse. Find its principal axes, and draw the ellipse on the x-y plane. [*Hint*: Note that

$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   Digonalize the 2x2 matrix in the middle of this quadratic form and perform a change of coordinates.]