EE2030 Linear Algebra
2nd Midterm Exam, Dec. 4, 2015
Lecturer: Yi-Wen Liu

Full score = 105. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way.
- **Mysterious or unsupported answers will not receive full credit**.

1. (20 points) **(System of linear equations).** Below, find the set of all solutions to $Ax = b$.

   (a) $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 1 & 0 & 2 & -2 \end{pmatrix}, b = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 100 \\ 260 \\ 2\pi \end{pmatrix}$.

2. (10 points) **(Vandermonde matrix).** Let $f(x) = a + bx + cx^2$ be a polynomial.

   (a) Assume that $f(1) = 1, f(2) = 0$ and $f(3) = 1$. Find $f(4)$.

   (b) Calculate the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 1 & 4 & 9 & 25 \\ 1 & 8 & 27 & 125 \end{vmatrix}$.

3. (30 points) **(Eigenvalue problem and change of coordinate)**. Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

   (a) (10 points) Let $\theta = \pi/6$. ($\therefore \cos\theta = \sqrt{3}/2$ and $\sin\theta = 1/2$). Find all of the eigenvalues of A.

   (b) (5 points) Show that $(1,1)^t$ and $(1,-1)^t$ are eigenvectors for A.

   (c) (5 points) Generally for any $\theta \in (0, \pi/4)$, describe what the corresponding linear transformation $\mathsf{L}_A : R^2 \to R^2$ does using a geometric illustration. In your opinions, does it make sense that $(1,1)^t$ and $(1,-1)^t$ are eigenvectors regardless of the choice of θ?

   (d) (5 points) Let $\beta = \{(1,1)^t, (1,-1)^t\}$ be an ordered basis for R^2, and Let γ be the standard basis, i.e., $\gamma = \{(1,0)^t, (0,1)^t\}$. Let I_2 be the identity transformation on R^2, i.e., $\mathsf{I}_2(v) = v$, for any $v \in R^2$. Calculate the change-of-coordinate matrix $Q = [\mathsf{I}_2]_\gamma^\beta$.

   (e) (5 points) Show that $[\mathsf{L}_A]_\beta$ is diagonal.

4. (10 points) **(Elementary operations)**. Let $A = \begin{pmatrix} 3 & 8 \\ 8 & 3 \end{pmatrix}$. Write A as a product of elementary matrices. [Hint: Find A^{-1} via row operations.]

5. (10 points) **(Orthonormal Group)**. Suppose that A and B are square matrices and $A^tA = I = B^tB$. Let $C = AB$.
   (a) Show that $C^tC = I$.
   (b) Show that $\det(A) = \pm 1$.

6. (25 points) **True or false**. For each statement below, if you think it is true, prove it. Otherwise, explain what is wrong. *Right answers with no explanation receive 2 points each.* Wrong answers with reasonable explanation will be considered partial credits.

   (a) Let
   $$M = \begin{pmatrix} O & A \\ I & B \end{pmatrix},$$
   where O is a zero matrix, I is an identity matrix, and A is a square matrix. Then $\det(M) = -\det(A)$.

   (b) Let A and B be square matrices of the same size. If $AB = I$, then $BA = I$.

   (c) Let $A \in \mathsf{M}_{m\times n}(R)$ and $B \in \mathsf{M}_{n\times m}(R)$. If $AB = I_m$, then $BA = I_n$.

   (d) Let A be an $n \times n$ matrix, and $b \in \mathsf{M}_{n\times 1}(R)$. If the system of equations $Ax = b$ has infinitely many solutions, then the last row of A can be written as a linear combination of other rows in A.

   (e) Let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear operator, and let β and β' be two ordered bases for V. Assume that V is finite-dimensional. Then, $\det([\mathsf{T}]_\beta) = \det\left([\mathsf{T}]_{\beta'}\right)$.