EE2030 Linear Algebra
1st Midterm Exam, Oct. 23, 2015
Lecturer: Yi-Wen Liu

This exam contains 2 pages and 6 problems. Check to see if any pages are missing.

The following rules apply:

- **Organize your work** in a reasonably neat and coherent way.
- **Mysterious or unsupported answers will not receive full credit**.

1. (15 points) Let v_1, v_2, v_3 be vectors in $\mathbb{R}^5$, and

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) (10 points) Show that $\{v_1, v_2, v_3\}$ are linearly dependent, and determine $\dim\big(\text{span}(\{v_1, v_2, v_3\})\big)$.

   (b) (5 points) Let $A = [v_1|v_2|v_3]$ be a 5×3 matrix, and let L_A denote the corresponding *left multiplication* transform from $\mathbb{R}^3$ to $\mathbb{R}^5$. Find a vector that belongs to $\mathsf{N}(L_A)$.

2. (10 points) Let $\mathsf{P}(R)$ denote the set of all polynomials with real coefficients. Define $\mathsf{T} : \mathsf{P}(R) \to \mathsf{P}(R)$ by

$$\mathsf{T}\big(f(x)\big) = \int_0^x f(t)dt.$$

   Show that T is one-to-one, but not onto.

3. (20 points) Let $\mathsf{T}_A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as follows:

$$\mathsf{T}_A(v) = A \times v \triangleq (A_y v_z - A_z v_y, A_z v_x - A_x v_z, A_x v_y - A_y v_x)^T,$$

   for any $v = (v_x, v_y, v_z)^T$ in $\mathbb{R}^3$, and let us assume that $A = (A_x, A_y, A_z)^T$ is fixed.

   (a) Show that T_A is a linear transformation.

   (b) Define a basis β for $\mathbb{R}^3$ (suggestion: use the most common definition) and write down $[\mathsf{T}_A]_\beta^\beta$.

   (c) If $A = (0, 0, 1)^T$, describe or make a sketch of $\mathsf{R}(\mathsf{T}_A)$.

   (d) Continuing from above, describe or make a sketch of $\mathsf{N}(\mathsf{T}_A)$.

4. (20 points) **Matrix multiplication.**
   (a) Let
   $$A = \begin{pmatrix} \cos\frac{2\pi}{67} & -\sin\frac{2\pi}{67} \\ \sin\frac{2\pi}{67} & \cos\frac{2\pi}{67} \end{pmatrix}$$
   be a 2×2 matrix. Argue that $A^{67} = I$.
   (b) For any square matrix $C \in \mathsf{M}_{n\times n}(\mathbb{R})$, define $f(C) = \sum_{i=1}^{n} C_{ii}$. In other words, $f(C)$ is the sum of the diagonal elements of C. Prove that, for any $A, B \in \mathsf{M}_{n\times n}(\mathbb{R})$, we have $f(AB) = f(BA)$.
   [*Remark*: $f(C)$ is called the trace of C and we will show later in this semester that the trace of a matrix equals to the sum of all its *eigenvalues*.]

5. (15 points) In no more than 5 lines, briefly answer each of the following questions.
   (a) At the beginning of this semester, we went through the 8 properties that a vector space must obey. One of them requires that $1x = x$ for each x in V. Could anything go wrong if a vector space does not require $1x = x$?
   (b) If a matrix represents a linear transformation, what does the product of two matrices represent?
   (c) What are the two properties that a basis for a vector space must have?

6. (20 points) **True or false**. For each statement below, if you think it is true, prove it. Otherwise, explain what is wrong. *Right answers with no explanation receive 2 points each.* Wrong answers with reasonable explanation will be considered partial credits.
   (a) Let $S_1 \subseteq S_2$ be two non-empty subsets of a vector space V. If $\text{span}(S_2) = \mathsf{V}$, then $\text{span}(S_1) = \mathsf{V}$.
   (b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation. If $\{v_1, ..., v_n\} \subseteq \mathsf{V}$ is linearly dependent, then $\{\mathsf{T}(v_1), ..., \mathsf{T}(v_n)\}$ is linearly dependent.
   (c) Continuing from above, if $\{v_1, ..., v_n\} \subseteq \mathsf{V}$ is linearly independent, then $\{\mathsf{T}(v_1), ..., \mathsf{T}(v_n)\}$ is linearly independent.
   (d) Let S be a subset of a vector space V. Then, $\text{span}(S)$ is the intersection of all subspaces of V that contain S.