EE2030 Linear Algebra
Midterm I, Oct. 29, 2014
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This exam contains 2 pages and 6 problems. Check to see if any pages are missing.

The following rules apply:

- **Organize your work** in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

1. (15 points) Briefly answer the following questions.
   (a) What is the definition of an *isomorphism*?
   (b) If a matrix represents a linear transformation, what does the product of two matrices represent?
   (c) Let $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be a function such that $\mathsf{T}(x) = x + (1, 0, 0)$ for any $x \in \mathbb{R}^3$. Show that T is not a linear transformation.

2. (15 points) Let v_1, v_2, v_3 be vectors in $\mathbb{R}^5$, and

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) (10 points) Determine $\dim(\text{span}(\{v_1, v_2, v_3\}))$.
   (b) (5 points) Let $A = [v_1|v_2|v_3]$ be a 5×3 matrix, and let L_A denote the corresponding *left multiplication* transform from $\mathbb{R}^3$ to $\mathbb{R}^5$. Find $\dim\big(\mathsf{N}(L_A)\big)$.

3. (10 points) Let $\mathsf{P}(R)$ denote the set of all polynomials with real coefficients. Define $\mathsf{T} : \mathsf{P}(R) \to \mathsf{P}(R)$ by

$$\mathsf{T}\big(f(x)\big) = \int_0^x f(t)dt.$$

   Show that T is linear, one-to-one, but not onto.

4. (10 points) Continuing from above, let V denotes the set of all *rational functions*, i.e.,

$$\mathsf{V} = \{f(x) = \frac{B(x)}{A(x)} : B(x), A(x) \in \mathsf{P}(R), A(x) \neq 0\}.$$

Do you think V is a vector space? Support your answer with clear arguments.

5. (25 points) Let V be the vector space of all real-valued function defined on $t \in (-\infty, \infty)$, and $\mathsf{W} = \text{span}(\{\cos t, \sin t\})$ is a subspace of V. Define $\mathsf{D} : \mathsf{W} \to \mathsf{W}$ as

$$\mathsf{D}(f(t)) = f(t - \tau).$$

for all $f(t) \in \mathsf{W}$.

(a) (10 points) Show that $\beta = \{\cos t, \sin t\}$ is linearly independent.

(b) (10 points) Write down the matrix representation $[\mathsf{D}]_\beta^\beta$.

(c) (5 points) Is $\mathsf{D} : \mathsf{W} \to \mathsf{W}$ onto?

6. (25 points) **True or false**. For each statement below, if you think it is true, prove it. Otherwise, explain what is wrong. *Right answers with no explanation receive 2 points each.* Wrong answers with reasonable explanation will be considered partial credits.

(a) Let S_1, S_2 be two subsets of a vector space V. If $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(b) Let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear transformation. If $\{v_1, ..., v_n\} \subseteq \mathsf{V}$ is linearly dependent, then $\{\mathsf{T}(v_1), ..., \mathsf{T}(v_n)\}$ is linearly dependent.

(c) Continuing from above, if $\{v_1, ..., v_n\} \subseteq \mathsf{V}$ is linearly independent, then $\{\mathsf{T}(v_1), ..., \mathsf{T}(v_n)\}$ is linearly independent.

(d) Let A and B be two matrices. If $AB = I$ is an identity matrix, then both A and B are invertible square matrices.

(e) There does not exist a linear transformation $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathsf{N}(\mathsf{T}) = \mathsf{R}(\mathsf{T})$.