EE2030 (Linear Algebra) Final Exam

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Brought to you by Yi-Wen Liu; full score = 105.

Part I. 是非簡答題 **(30%).**

1. Please mark **T**rue or **F**alse for each of the following statements. If False, point out where it goes wrong. If True, give an explanation as concisely as possible.

- Each correct answer of T or F is guaranteed to get 1 point
- Incorrect answer with reasonable explanation gets 3 points at most.
- Only a correct answer with decent/accurate explanation gets full score (5 point each).
- 課本上已有的證明都可以直接引用。

(a) If an $n \times n$ matrix is diagonalizable, it is invertible.

(b) If $A \in M_{n\times n}(\mathbb{C})$ has n distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, then $\det(A) = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n$.

(c) Let S be a subset of an inner product space V. Then $(S^{\perp})^{\perp} \supseteq \text{span}(S)$.

(d) Let T be a linear operator over an inner-product space V. If $\mathsf{T}(x) = \lambda x$, and $\mathsf{T}^*(y) = \mu y$, then $\lambda = \bar{\mu}$.

(e) Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. Assume that $x \neq 0$. Then x is an eigenvector for the matrix $B = xx^t \in M_{3\times 3}(\mathbb{R})$, and the corresponding eigenvalue is positive.

(f) Let T be a linear operator over a finite dimensional inner-product space V. Assume that x_1 and x_2 are two eigenvectors of T associated with the same eigenvalue, i.e., assume that $\mathsf{T}(x_1) = \lambda x_1$, $\mathsf{T}(x_2) = \lambda x_2$. Under these assumptions, if $\langle x_1, x_2 \rangle = 0$, then λ is a multiple root of the characteristic polynomial of T.

Part II. 簡答題（**25%**）

2. (10 pts) **Least square problems**. Let $b \in M_{m\times 1}(\mathbb{R})$ and $A \in M_{m\times n}(\mathbb{R})$. Describe the conditions for a unique x to exist such that it minimizes $\|Ax - b\|^2$.

3. (15 pts) Let $X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathbb{R}^2$, where t denotes the continuous time variable. Let $A \in M_{2\times 2}(\mathbb{R})$ be an arbitrary matrix that does not change in time. Describe a general way to solve the set of differential equations $\dot{X}(t) = AX(t)$, given the initial condition $X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

Part III: 計算題（50%）

4. (25 pts) **Schur Decomposition**. Let $A = \begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}$.

(a) [10%] Calculate the eigenvalues and the corresponding eigenspaces for A.

(b) [3%] Does A^t have the same eigenvalues?

(c) [3%] Find a vector z_1 such that z_1 is an eigenvector of A^t and $\|z_1\| = 1$.

(d) [3%] Find a vector z_2 such that $z_2 \in z_1^{\perp}$ and $\|z_2\| = 1$.

(e) [3%] Let $Q = [z_2|z_1] \in M_{2\times 2}(\mathbb{R})$. Show that $Q^t = Q^{-1}$.

(f) [3%] Argue that $AQ = QU$, where U is an upper triangular matrix.

5. (25%) Let $S = \{f_1(t), \dots, f_n(t)\}$ be a set of functions defined on $t \in [-\pi, \pi]$, and let $f_k(t) = \cos kt$, $k = 1,2, \dots, n$. For any pair of integrable （可積分的） real functions $x(t)$ and $y(t)$ on $[-\pi, \pi]$, define inner product $\langle x(t), y(t) \rangle = \int_0^{2\pi} x(t)y(t)dt$.

(a) [5%] Show that $\langle f_i(t), f_j(t) \rangle = 0, \forall i \neq j$.

(b) [5%] Calculate $\|f_k(t)\|$ for $k = 1,2, \dots, n$.

(c) [5%] Let $y(t)$ be defined on $[-\pi, \pi]$ as follows,

$$y(t) = \begin{cases} 1, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\langle y(t), f_j(t) \rangle$ for $j = 1, 2, 3$.

(d) [5%] Find the function $u(t) \in \text{span}(S)$ such that $\langle y - u, x \rangle = 0$ for all $x \in \text{span}(S)$.

(e) [5%] Express $\|y - u\|^2$ in terms of n.