Unit 7.1 Backtracking

Algorithms

EE3980

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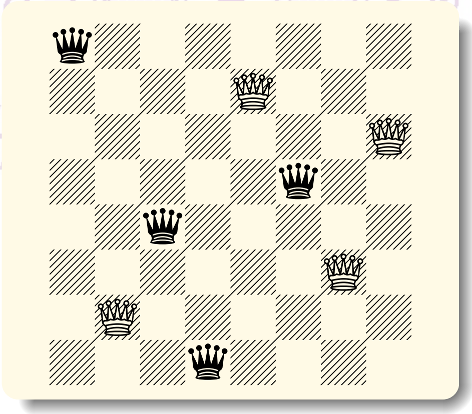
Backtracking Algorithms

- The backtracking algorithms are to deal with problems to generate a desired solution expressible as an n-tuple, $(x_1, x_2, \cdots, x_n)$, where x_i are chosen from a finite set S_i, and the solution satisfies or minimizes/maximizes a criterion function $P(x_1, x_2, \cdots, x_n)$.
- Suppose m_i is the size of S_i. Then, there are $m = m_1 \times m_2 \times \cdots \times m_n$ possible candidates for satisfying the function P.
- The brute force approach is to form all m candidates and evaluate criterion function on each of them, selects all (or the optimal) solutions.
- The backtracking method form the n-tuple one component at a time, and then use the modified criterion function $P_i(x_1, x_2, \cdots, x_i)$ to see if the current vector can meet the overall criterion. If it cannot, the current vector is ignored immediately.
  - The number of tries is substantially smaller with backtracking methods.

The 8-Queens Problem

- A queen in a chess game can attack any other piece if
  - It is in the same row
  - It is in the same column
  - It is in the same diagonal (two directions)
- The 8-queens puzzle is to place 8 queens on a chessboard such that they don't attack each other.



The N-Queens Problem — Algorithms

- The problem is generalized to placing N-queens onto an $N \times N$ board.
- Note that each row can have only one queen.
  - Thus, one can use an array $Q[1:N]$ for column position for each queen.
- For 8-Queens puzzle this reduces the number of possible checks from 8^8 (16,777,216), to $8\,!$ (40,320), 0.24%.

Algorithm 7.1.1. N-Queen puzzle

```
// Place k'th queen onward, if successful print out solution.
// Input: Q[1 : n], k, n; Output: Solutions for placing n-Queens.
Algorithm NQueens(k, n)
{
    for i := 1 to n do { // all possible positions for Q[k]
        if Placeable(k, i) then { // placing Q[k] = i is legitimate
            Q[k] := i;
            if (k = n) then write (Q[1 : n]); // a solution found.
            else NQueens(k + 1, n); // place Q[k + 1]
        }
    }
}
```

The N-Queens Problem — Algorithms, II

- The Placeable algorithm is shown below.

Algorithm 7.1.2. Placeable

```
// Test if it is legitimate to place a Queen at (k, i).
// Input: Q[1 : k], i; Output: 1: if OK to place, 0: otherwise.
Algorithm Placeable(k, i)
{
    for j := 1 to k − 1 do { // check against Queens placed.
        if (Q[j] = i or |Q[j] − i| = |j − k|) then return false ;
    }
    return true ;
}
```

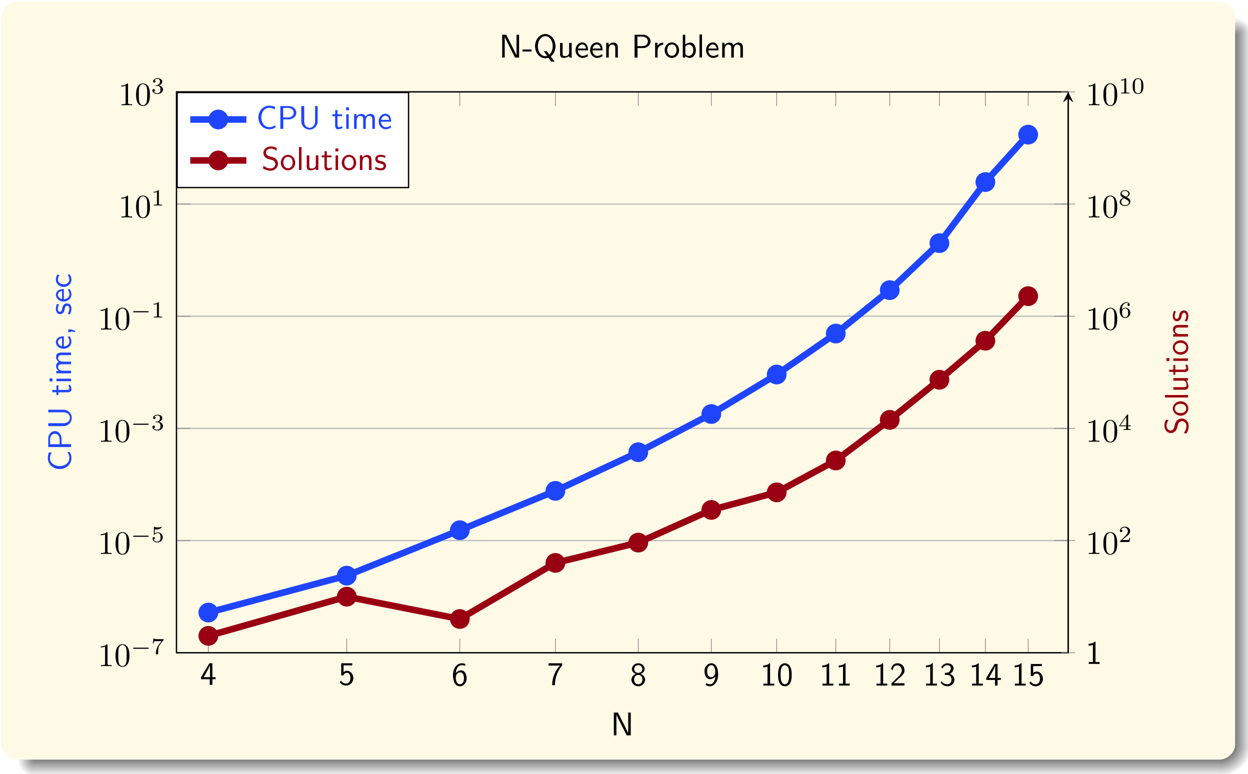
- Note that if a queen is not placeable to (k, i) then rows k onwards are not checked, thus reducing even more checkings.
- An iterative version, NQueens_I, is given next
  - Same time complexity as the recursive version
  - The number of try-outs is identical in either case
  - Smaller heap space for function calls for iterative version
- Both NQueens and NQueens_I algorithms can still be improved.

The N-Queens Problem — Iterative Algorithms

Algorithm 7.1.3. N-Queen puzzle, iterative solution

```
// Find all solutions for N-Queen problem iteratively.
// Input: number of Queens: n; Output: Solutions for placing n-Queens.
Algorithm NQueens_I(n)
{
    k := 1; Q[k] := 0;
    while (k > 0) do {
        Q[k] := Q[k] + 1;
        while (Q[k] ≤ n) do {
            if Placeable(k, Q[k]) then {
                if (k = n) then write (Q[1 : n]); // a solution is found
                else {
                    k := k + 1; Q[k] := 0; // try next row and initialize
                }
            }
            Q[k] := Q[k] + 1;
        }
        k := k − 1; // done with this row, backtrack to previous row
    }
}
```

The N-Queen Problem – Solutions



- Both CPU time and number of solution appear to increase exponentially with N.
- The complexity of the algorithms, recursive and iterative, are $\mathcal{O}(N!)$.

Sum of Subsets Problem

- Given a set of n distinct positive numbers, $\{w_i, 1 \le i \le n\}$ and m, $m > 0$, the sum of subsets problem is to find all the combinations of those n numbers whose sum is m.
- Example, given the set $\{4, 11, 15, 24\}$ and $m = 15$.
  - Two subsets, $\{4, 11\}$ and $\{15\}$, have the sum equals to 15.
- It is assume that the set is ordered in nondecreasing order, $w_i \le w_{i+1}, 1 \le i < n$, and

$$w_1 \le m, \tag{7.1.1}$$

$$\sum_{i=1}^{n} w_i \ge m. \tag{7.1.2}$$

  otherwise, there is no solution possible.
- Let $\{x_i | x_i = 0 \text{ or } x_i = 1, 1 \le i \le n\}$ be the solution, then

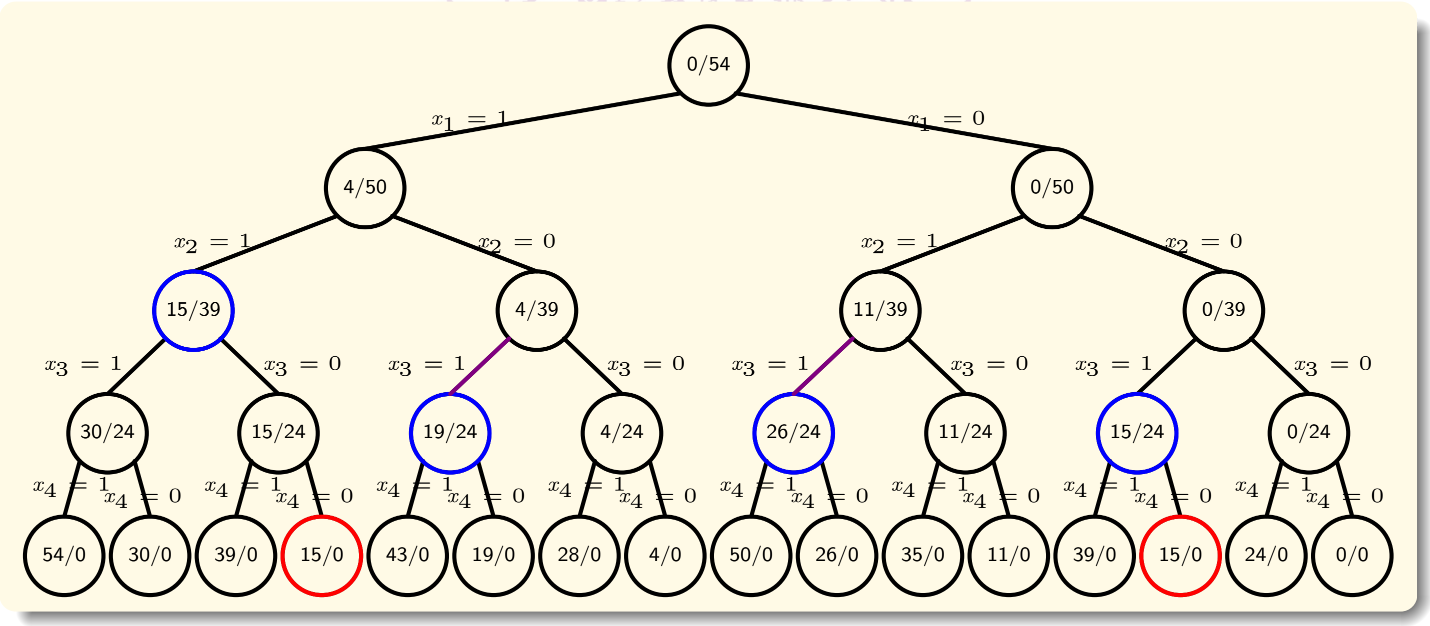
$$\sum_{i=1}^{n} x_i w_i = m. \tag{7.1.3}$$

- To find the solution, all combinations are to be tested.
  - Backtracking approach can be applied.

Sum of Subsets, Example

- Example, $w = \{4, 11, 15, 24\}$, $m = 15$
- Two numbers shown in each node s/r

$$s = \sum_{i=1}^{k} x_k w_k \qquad r = \sum_{i=k+1}^{n} w_k$$



Sum of Subsets, Algorithm

- A recursive algorithm to find all the solutions.

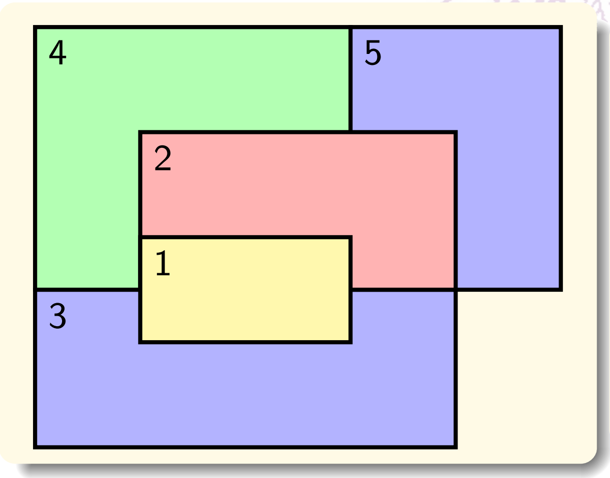
Algorithm 7.1.4. Sum of Subsets

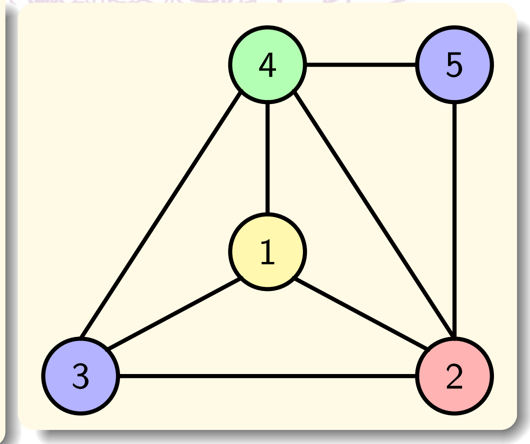
```
    // To test if x[k] = 1 for sum of subset problem.
    // Input: s = Σ_{i=1}^{k-1} w[i]x[i], r = Σ_{i=k}^{n} w[i], k, w[1 : n] ; Output: x[1 : n].
 1 Algorithm SumOfSub(s, k, r)
 2 {
 3     x[k] := 1 ; // try to include w[i]
 4     if (s + w[k] = m) then write (x[1 : k]) ; // one solution found
 5     else if (s + w[k] + w[k + 1] ≤ m) then
 6         SumOfSub(s + w[k], k + 1, r − w[k]) ;
 7     if ((s + r − w[k] ≥ m) and (s + w[k + 1] ≤ m)) then { // x[i] = 0 case
 8         x[k] := 0 ;
 9         SumOfSub(s, k + 1, r − w[k]) ;
10     }
11 }
```

- The definition of s is different from the preceding page.
- Note the termination condition of this algorithm
- With proper checking, lines 5 and 7, number of unsuccessful search is significantly reduced, but the complexity remains to be $\mathcal{O}(2^n)$.

Graph Coloring

- Given a map with n regions, the m-colorability decision problem is to find if one can assign m colors to the map such that each region has a color and no two adjacent regions have the same color.
- Note that the map with n regions can be transformed into a graph.
  - Each region is represented by a node,
  - Adjacent regions are connected by an edge between the nodes.
- The adjacency relationship can also be represented by the adjacency matrix.





$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Graph Coloring, Algorithm

- The following algorithm solves for the m-coloring problem for a graph, G, with n vertices and adjacency matrix A.
  - Global Array x is the solution found, $x[i]$ is the color for vertex i.
- The algorithm should be invoked by
  mColoring$(n, m, 1)$;

Algorithm 7.1.5. m-Color Algorithm

```
// Recursively assign all possible, at most m, colors to node k.
// Input: int n, m, k, adjacent matrix A ; Output: All acceptable solutions.
Algorithm mColoring(n, m, k)
{
    for x[k] := 1 to m do {
        i := 1 ; // check for colored and adjacent nodes with the same color
        while (i < k and ((A[i, k] = 0) or (x[i] ≠ x[k]))) do i := i + 1 ;
        if (i = k) then { // color acceptable
            if (k = n) then write (x[1 : n]) ; // a solution is found
            else mColoring(n, m, k + 1);
        }
    }
}
```

Graph Coloring, Complexity

- In algorithm mColoring, Algorithm (7.1.5), the for loop, lines 3-10, is executed m times at each recursive call
  - And mColoring is called recursively for n times
- Again in algorithm mColoring, the while loop, line 5 is executed at most n times
- Thus the total time complexity is $\mathcal{O}(nm^n)$
- An alternative algorithm is shown on the next page.
  - The color to be tested for node k is reduced – no repeated colors.
  - But the complexity is $\mathcal{O}(m^n)$.

- Note that given a graph G with degree d, then G can be colored using $d+1$ colors.
- The smallest m that can color a graph G is also called the chromatic number of G.
- Note that $m \leq d+1$ and m can be found by using the Algorithm mColoring using different m.

Graph Coloring, An Alternative Algorithm

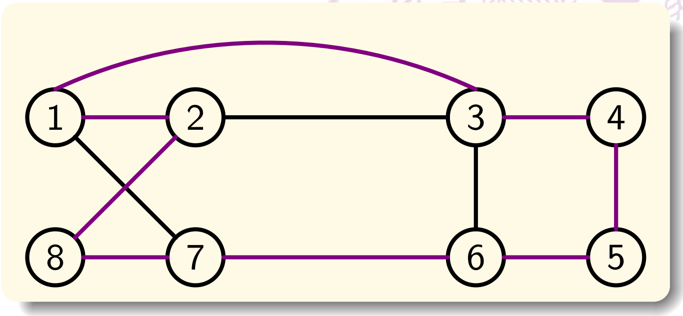
- An alternative algorithm for the m-coloring problem.

Algorithm 7.1.6. m-Color Algorithm – Alternative

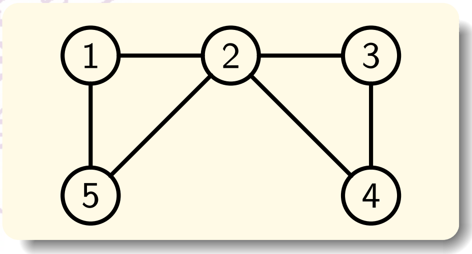
```
// Recursively assign all possible, at most m, colors to node k.
// Input: int n, m, k, adjacent matrix A ; Output: All acceptable solutions.
Algorithm mColoring_A(n, m, k)
{
    for i := 1 to m do c[i] := 1 ; // Let all colors be available.
    for i := 1 to k − 1 do
        if A[i, k] = 1 then c[x[i]] := 0 ; // Color used by adj. nodes.
    for i := 1 to m do {
        if c[i] = 1 then { // Use all available colors.
            x[k] := i ;
            if (k = n) then write (x[1 : n]) ; // A solution is found
            else mColoring_A(n, m, k + 1) ;
        }
    }
}
```

Hamiltonian Cycles

- Let $G = (V, E)$ be a connected graph with n vertices. A Hamiltonian cycle is a closed path along n edges of G that visits every vertex once and returning to its starting position.
  - If a Hamiltonian cycle begin at a vertex $v_1 \in V$ and the vertices are visited in the order $(v_1, v_2, \cdots, v_{n+1})$, then the edge $(v_i, v_{i+1}) \in E$, $1 \le i \le n$, and the v_i are distinct except $v_1 = v_{n+1}$.



G_1 with Hamiltonian cycles.



G_2 No Hamiltonian cycle.

Hamiltonian Cycles — Algorithm

Algorithm 7.1.7. Hamiltonian Cycle

```
   // Recursively find the k-th vertex of a Hamiltonian cycle.
   // Input: graph G(V, E), int n, k; Output: All possible Hamiltonian cycles.
Algorithm Hamiltonian(n, k)
{
    for x[k] := 1 to n do { // All possible vertices.
        if (E[x[k − 1], x[k]] = 1) then { // Connecting to x[k − 1].
            i := 1 ;
            while ((i < k) and (x[i] ≠ x[k])) i := i + 1 ; // Check if x[k] distinct
            if (i = k) then // x[k] has not been used
                if (k < n) Hamiltonian(n, k + 1) ; // Move to the next vertex
                else {
                    if (E[k, 1] = 1) then write (x[1 : n]) ; // Print solution
                }
        }
    }
}
```

Hamiltonian Cycles — Algorithm, II

- Backtracking approach to solve the Hamiltonian cycle problem.
- $x[1:n]$ is the solution vector.
- $E[1:n, 1:n]$ is the adjacency matrix
  - $E[i,j] = 1$ if $(i,j) \in E$ is an edge in G
  - Otherwise, $E[i,j] = 0$.
- `Hamiltonian` should be invoked by
  `Hamiltonian`$(n, 2)$;
  with $x[1] = 1$.
- Thus, this algorithm always find the Hamiltonian cycle starting from vertex 1.
- Note that the depth of the recursive call is n
  - The maximum number of `Hamiltonian` recursive call at level k is $n-k$ since each vertex on the path must be distinct
  - Thus, the number of function call is bounded above by $(n-1)!$
- The `while` loop of line 6 is executed at most n times
- The worst case time complexity of `Hamiltonian` algorithm is $\mathcal{O}(n!)$
  - Due to the sparsity of the adjacency matrix, this algorithm has much lower complexity in practice.

0/1 Knapsack Problem

- Given n objects, each with profit p_i and weight w_i, $1 \le i \le n$, to be placed into a sack that can hold maximum of m weight. However, there is an additional constraint that each object must be placed as a whole into the sack, or not at all. That is, find x_i, $1 \le i \le n$, such that

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} p_i x_i, \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i x_i \le m, \\ \text{and} \quad & x_i = 0 \text{ or } 1, \qquad 1 \le i \le n. \end{aligned} \tag{7.1.4}$$

- Note that $x_i = 0$ or 1 and the solution space can be expanded as a tree.
- The solution can be found by traversing the tree.
- In the following, we assume the objects are ordered as

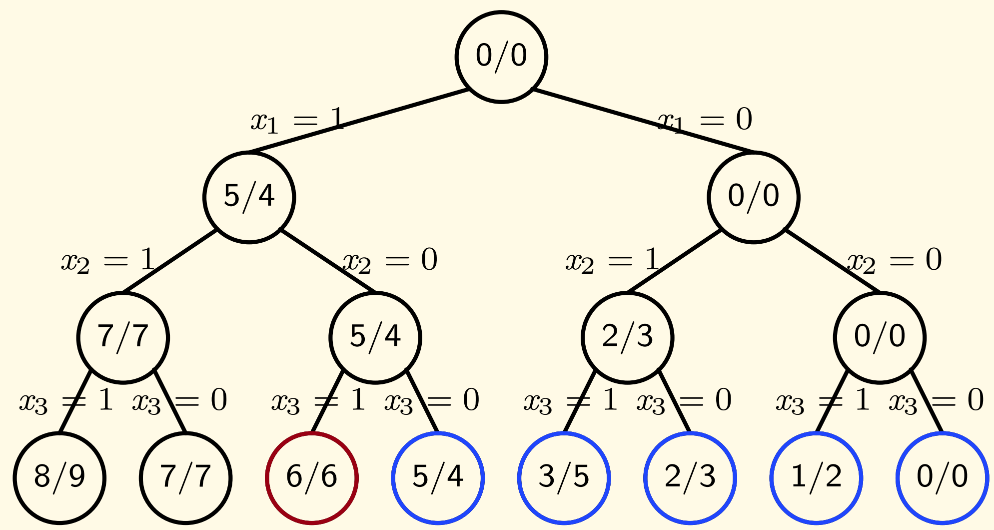
$$\frac{p_1}{w_1} \ge \frac{p_2}{w_2} \ge \cdots \ge \frac{p_n}{w_n}. \tag{7.1.5}$$

  And, fp is the final profit, fw is the final weight. Both are global variables.

0/1 Knapsack Problem — Search Space

- Given 3 objects, $(p_1, p_2, p_3) = (5, 2, 1)$, $(w_1, w_2, w_3) = (4, 3, 2)$, and $m = 6$. Find the optimal 0/1 knapsack solution, (x_1, x_2, x_3), $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit.
- Two numbers shown in each node cp/cw

$$cp = \sum_{i=1}^{k} x_k p_k \qquad cw = \sum_{i=1}^{k} x_k w_k$$



Red: optimal solution; Blue: feasible solutions

0/1 Knapsack Problem — Algorithm

Algorithm 7.1.8. 0/1 Knapsack

```
    // Find solution of 0/1 knapsack problem.
    // Input: int k, n, cp/cw/cx: current profit/weight/sol ; Output: Solution x[1 : n].
Algorithm BKnap(k, cp, cw)
{
      if (cw + w[k] ≤ m) then { // Add k-th object.
            cx[k] := 1 ;
            if (k < n) then BKnap(k + 1, cp + p[k], cw + w[k]) ; // Check next.
            else if ((cp + p[k] > fp) and (k = n)) then { // Record solution
                  fp := cp + p[k] ; fw := cw + w[k] ;
                  for i := 1 to n do x[i] := cx[i] ;
            }
      }
      if ( Bound(cp, cw, k) ≥ fp) { // Continue traversing only if needs to.
            cx[k] := 0 ; // Not placing k-th object.
            if (k < n) then BKnap(k + 1, cp, cw) ; // Check next object.
            else if ((cp > fp) and (k = n)) then { // Record solution.
                  fp := cp ; fw := cw ;
                  for i := 1 to n do x[i] := cx[i] ;
            }
      }
}
```

0/1 Knapsack Problem — Bound Algorithm

- Due to Eq. (7.1.5), Bound function can estimate the maximum profit quickly.

Algorithm 7.1.9. Bounding function

```
   // Estimate maximum profit for k+1 to n objects.
   // Input: int k, n, cp/cw: current profit/weight ; Output: maximum profit mp.
Algorithm Bound(cp, cw, k)
{
    mp := cp ; mw := cw ; // Init to current values.
    for i := k+1 to n do { // Evaluate all possible.
        mw := mw + w[i] ; // Update maximum weight.
        if (mw < m) then mp := mp + p[i] ; // Within limit.
        else return mp + (1 - (mw - m)/w[i]) * p[i] ; // Exceeding limit.
    }
    return mp ;
}
```

- Note that Bound function returns a floating number instead of an integer.

0/1 Knapsack Problem — Example

- Given 3 objects, $(p_1, p_2, p_3) = (5, 2, 1)$, $(w_1, w_2, w_3) = (4, 3, 2)$, and $m = 6$. Find the optimal 0/1 knapsack solution, (x_1, x_2, x_3), $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit.
- The calling sequence of BKnap algorithm

BKnap($k = 1$, $cp = 0$, $cw = 0$)
    test $cx[1] = 1$, $cw + w[1] \le m$
    BKnap($k = 2$, $cp = 5$, $cw = 4$)
        test $cx[2] = 1$, $cw + w[2] > m$
        test $cx[2] = 0$, Bound$= 6$
        BKnap($k = 3$, $cp = 5$, $cw = 4$)
            test $cx[3] = 1$, $cw + w[3] = m$, feasible solution: $fp = 6$, $x = (1, 0, 1)$
            test $cx[3] = 0$, Bound$= 5$, terminates
    test $cx[1] = 0$, Bound$= 3$, terminates

- Function Bound helps to reduce the number of evaluations

Summary

- Backtracking algorithm
- 8-queens problem
- Sum of subsets problem
- Graph coloring problem
- Hamiltonian cycles
- 0/1 knapsack problem