# Unit 6.3 Dynamic Programming III

Algorithms

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## String Editing Problem

- Given two strings $X =$"$x_1x_2\cdots x_n$" and $Y =$"$y_1y_2\cdots y_m$", where $x_i$, $1 \le i \le n$, and $y_j$, $1 \le j \le m$, are members of a finite set of symbols known as the alphabet.
- The string editing problem is to transform $X$ into $Y$ using the following editing operations with corresponding cost and to find the sequence of operations that minimizes the total cost.
  - Delete the symbol $x_i$ from $X$ with cost $D(x_i)$,
  - Insert the symbol $y_j$ to $Y$ with cost $I(y_j)$,
  - Change the symbol $x_i$ of $X$ into $y_j$ with cost $C(x_i, y_j)$.
  - Note that keep $x_i$ to become $y_j$ has no cost.
- Example, $X =$"$elate$" and $Y =$"$later$". Total cost to transform $X$ into $Y$ is $D(e) + I(r)$.

| Step | $X$ | $Y$ | Cost |
|---|---|---|---|
| 1 | $elate$ | | $D(e)$ |
| 2 | $elate$ | $l$ | 0 |
| 3 | $elate$ | $la$ | 0 |
| 4 | $elate$ | $lat$ | 0 |
| 5 | $elate$ | $late$ | 0 |
| 6 | $elate$ | $later$ | $I(r)$ |
| | | | $D(e) + I(r)$ |

# String Editing — Algorithm

## Algorithm 6.3.1. Wagner Fischer Algorithm

```
// Transform X into Y with minimum cost using matrix M.
// Input: int n, m, strings X, Y, cost D, I, C; Output: min cost matrix M.
Algorithm WagnerFischer(n, m, X, Y, D, I, C, M)
{
    M[0,0] := 0;
    for i := 1 to n do M[i,0] := M[i-1,0] + D(X[i]);
    for j := 1 to m do M[0,j] := M[0,j-1] + I(Y[j]);
    for i := 1 to n do {
        for j := 1 to m do {
            if (X[i] = Y[j]) then m1 := 0; else m1 := C(X[i], Y[j]);
            m2 := M[i-1,j] + D(X[i]);
            m3 := M[i,j-1] + I(Y[j]);
            M[i,j] := min(m1, m2, m3);
        }
    } // When done, M[n,m] contains the minimum cost of the transformation
}
```



# String Editing — Example

- Example. Given $X =$ "*elate*", $Y =$ "*later*" and the cost functions $D(x) = 1,\ I(y) = 1,\ C(x, y) = 2,\ x, y \in \{A, \cdots, Z, a, \cdots, z\},\ x \neq y$.

| | | $l$ | $a$ | $t$ | $e$ | $r$ |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| $e$ | 1 | 2 | 3 | 4 | 3 | 4 |
| $l$ | 2 | 1 | 2 | 3 | 4 | 5 |
| $a$ | 3 | 2 | 1 | 2 | 3 | 4 |
| $t$ | 4 | 3 | 2 | 1 | 2 | 3 |
| $e$ | 5 | 4 | 3 | 2 | 1 | 2 |

Matrix $M$ of `WagnerFischer` algorithm.

- Thus the transformation sequence is

| Step | operation | | $Y$ |
|---|---|---|---|
| 1 | Delete | $e$ | |
| 2 | Keep | $l$ | $l$ |
| 3 | Keep | $a$ | $la$ |
| 4 | Keep | $t$ | $lat$ |
| 5 | Keep | $e$ | $late$ |
| 6 | Insert | $r$ | $later$ |

- And the total cost is $D(e) + I(r) = 2$.

- After `WagnerFischer` algorithm, the following algorithm traces the $M$ matrix to generate the transformation sequence.
  - Note that array $T$ has the transformation sequence but is in reverse order.

# String Editing — Transformation Trace

## Algorithm 6.3.2. Trace

```
   // Trace the matrix M to find the transformation operations.
   // Input: int n, m, cost D, I, C and M; Output: T transformation.
1  Algorithm Trace(n, m, M, D, I, C, T)
2  {
3      i := n; j := m; k := 0;
4      while (i > 0 or j > 0) do {
5          if (M[i, j] = M[i − 1, j − 1]) then { // Keep X[i] for Y[j].
6              T[k] := ' − '; i := i − 1; j := j − 1; k := k + 1;
7          }
8          else if (M[i, j] = M[i − 1, j − 1] + C(X[i], Y[j])) then { // Change.
9              T[k] := 'C'; i := i − 1; j := j − 1; k := k + 1;
10         }
11         else if (i = 0 or (M[i, j] = M[i − 1, j] + D(X[i]))) then { // Delete.
12             T[k] := 'D'; i := i − 1; k := k + 1;
13         }
14         else { // Add Y[j].
15             T[k] := 'I'; j := j − 1; k := k + 1;
16         }
17     } // Array T has the transformation sequence but is in reverse order.
18 }
```



# String Editing — Complexities

- Algorithm WagnerFischer
  - for loop, lines 6–13, executes $n \times m$ times
  - for loops, lines 6,7, execute $n$ and $m$ times, separately
  - Overall time complexity $\mathcal{O}(mn)$
- Algorithm Trace while loop, lines 4-17, executes at most $(m + n)$ times
  - Time complexity $\mathcal{O}(m + n)$
- The longest common substring problem
  - Given two strings, $X$ and $Y$, find a common substring $Z$ such that $Z$ has the most number of characters.
  - Example, $X =$"elate" and $Y =$"later" the longest common substring is $Z =$"late". $Z$ has 4 characters.
  - The WagnerFischer algorithm can be used to find the longest common substring.
  - The Trace algorithm needs to be modified to find and print out the common substring.

# 0/1 Knapsack Problem

- The 0/1 knapsack problem is a variation of the knapsack problem.
  - Given $n$ objects, each with profit $p_i$ and weight $w_i$, $1 \le i \le n$, to be placed into a sack that can hold maximum of $m$ weight. However, there is an additional constraint that each object must be placed as a whole into the sack, or not at all. That is, find $x_i$, $1 \le i \le n$, such that

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} p_i x_i, \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i x_i \le m, \\ \text{and} \quad & x_i = 0 \text{ or } 1, \qquad 1 \le i \le n. \end{aligned} \tag{6.3.1}$$

- Let $f_n(m)$ be the optimal solution to $n$-object 0/1 knapsack problem.
- For the $n$'th object it can either be placed into the sack or not, thus

$$f_n(m) = \max\Big(f_{n-1}(m), f_{n-1}(m - w_n) + p_n\Big). \tag{6.3.2}$$

  - $f_n(m)$ must be the larger of the following two cases
  - $n$-th object is not placed into the sack, $x_n = 0$,
    - In this case, $f_n(m) = f_{n-1}(m)$.
  - $n$-th object is placed into the sack, $x_n = 1$,
    - In this case, $f_n(m) = f_{n-1}(m - w_n) + p_n$.



# 0/1 Knapsack — Recursive Algorithm

- Using Eq. (6.3.2) a recursive version of the 0/1 knapsack algorithm can be formulated.

## Algorithm 6.3.3. Recursive DKP

```
// Find the solution array x for the 0/1 knapsack problem.
// Input: int n, profit p, weight w, m; Output: Solution x.
Algorithm DKPr(n, p, w, m, x)
{
    if (n = 1) then {
        if (m ≥ w[1]) then {
            x[1] := 1; return p[1];
        }
        else {
            x[1] := 0; return 0;
        }
    }
    f1 := DKPr(n − 1, p, w, m, x); // object n not placed
    if (m ≥ w[n]) then // placing n'th object
        f2 := DKPr(n − 1, p, w, m − w[n], x) + p[n];
    else f2 := 0; // no room for additional objects
    if (f1 > f2) then {
        x[n] := 0; return f1;
    }
    else {
        x[n] := 1; return f2;
    }
}
```

# 0/1 Knapsack — Example

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, $(x_1, x_2, x_3)$, $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit,

$$P = \sum_{i=1}^{3} p_i x_i.$$

- The function DKPr is invoked by calling
  $P =$DKPr$(3, p, w, 6, x)$
  - And the calling sequence of the function is

```
// DKPr calling sequence
DKPr(3, p, w, 6, x)
        DKPr(2, p, w, 6, x) // object 3 not placed
               DKPr(1, p, w, 6, x) // object 2 not placed
                      P = 1 ; x = (1, 0, 0) ;
               DKPr(1, p, w, 3, x) // object 2 placed
                      P = 3 ; x = (1, 1, 0) ;
        DKPr(2, p, w, 2, x) // object 3 placed
               DKPr(1, p, w, 2, x) // object 2 not placed
                      P = 6 ; x = (1, 0, 1) ;
               DKPr(1, p, w, −1, x) // object 2 placed
                      P = −∞ ; x = (0, 1, 1) ;
Maximum profit P = 6, x = (1, 0, 1).
```



# 0/1 Knapsack — Complexity

- Note that function DKPr is invoked 7 times
  - All possible combinations of $x_i = 0$ and $x_i = 1$, $1 \le i \le n$ are tested for the maximum profit.
- The time complexity of DKPr algorithm is $\mathcal{O}(2^n)$.
- Line 11 of DKPr algorithm can eliminate unnecessary function calls
  - If there is no room for object $n$ then it is not necessary to call DKPr further.
- The worst-case complexity of DKPr remains as $\mathcal{O}(2^n)$.

# 0/1 Knapsack — Dynamic Programming Approach

## Algorithm 6.3.4. 0/1 Knapsack

```
    // Find the solution array x for the 0/1 knapsack problem.
    // Input: int n, profit p, weight w, m; Output: Solution x.
Algorithm DKP(n, p, w, m, x)
{
      S_0^1 := {(0,0)};
      for i := 1 to n − 1 do {
            S_1^i := {(p + p_i, w + w_i)|(p, w) ∈ S_0^i and w + w_i ≤ m};
            S_0^{i+1} := MergePurge(S_0^i, S_1^i);
      }
      (px, wx) := last pair in S_0^n;
      (py, wy) := (p′ + p_n, w′ + w_n) where w′ is the largest w′ for any pairs
            (p′, w′) ∈ S_0^n such that w′ + w_n ≤ m;
      if (px > py) then x_n := 0;
      else x_n := 1;
      TraceBack x_{n−1}, · · · , x_1;
}
```



# 0/1 Knapsack — Example Revisited

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, $(x_1, x_2, x_3)$, $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit, $P = \sum_{i=1}^{3} p_i x_i$ .
- The sets of feasible solutions are derived as the following.

$$
\begin{aligned}
S_0^1 &= \{(0, 0)\} \\
S_1^1 &= \{(1, 2)\} \\
S_0^2 &= \{(0, 0), (1, 2)\} \\
S_1^2 &= \{(2, 3), (3, 5)\} \\
S_0^3 &= \{(0, 0), (1, 2), (2, 3), (3, 5)\}
\end{aligned}
$$

- The last pair in $S^2$ is $(p_x, p_y) = (3, 5)$, and $(p_y, w_y) = (6, 6)$.
- Thus the optimal solution $\sum p_i x_i = 6$ and $\sum w_i x_i = 6$.
  - Since $p_x \not> p_y$, $x_3 = 1$.
  - Note that $(p_y, w_y) - (5, 4) = (1, 2) \notin S_1^1$, thus $x_2 = 0$.
  - Trace back again, $(1, 2) \in S_1^0$, therefore $x_1 = 1$.
  - Finally we have $(x_1, x_2, x_3) = (1, 0, 1)$ and $\sum p_i x_i = 6$, $\sum w_i x_i = 6$.

# 0/1 Knapsack — Properties

- Note that lines 9, 10 of Algorithm (6.3.4) actually requires to evaluate $S_1^n$.
- For the last example, we have

$$S_1^3 = \{(5,4),(6,6)\}.$$

since $(7,7)$ and $(8,9)$ both have $w + w_n \not\leq m$.

- And the optimal solution can be found when $S_0^3$ and $S_1^3$ are merged together which is

$$S_0^4 = \{(0,0)(1,2)(2,3),(3,5),(5,4),(6,6)\}.$$

- Note that comparing $(3,5)$ and $(5,4)$, the former has smaller profit, $3 < 5$, but larger weight, $5 > 4$, thus it is not a likely solution.
- The former, $(3,5)$, is dominated by the latter, $(5,4)$.
- When merging two feasible sets, the dominated solutions should be purged.
- Of course, by definition, the solutions with weight larger than $m$ are also purged.



# 0/1 Knapsack — Dynamic Algorithm

## Algorithm 6.3.5. 0/1 Knapsack

```
struct PW {
      double p, w; // for profit and weight of each object
}
Algorithm DKnap(n, p, w, x, m)
// p and w are arrays of n profits and weight; m capacity, x solution.
{
      b[0] := 0 ; pair[1].p := 0 ; pair[1].w := 0 ; // S_0^1
      t := 1 ; h := 1 ; // start and end of S_0^1
      b[1] := next := 2 ; // next free spot in pair array
      for i := 1 to n do { // generate S_0^{i+1}
            k := t ;
            u := Largest(pair, t, h, w[i], m) ; // largest u, pair[u].w + w[i] ≤ m.
            for j := t to u do { // generate S_1^i and merge
                  pp := pair[j].p + p[i] ; ww := pair[j].w + w[i] ;
                  while ((k ≤ h) and (pair[k].w ≤ ww)) do {
                        pair[next].p := pair[k].p ; pair[next].w := pair[k].w ;
                        next := next + 1 ; k := k + 1 ;
                  }
                  if ((k ≤ h) and (pair[k].w = ww)) then {
                        if (pp < pair[k].p) then pp := pair[k].p ; // new entry dominated
                        k := k + 1 ;
                  }
                  if (pp > pair[next − 1].p) then { // new entry is dominating
                        pair[next].p := pp ; pair[next].w := ww ;
                        next := next + 1 ;
                  }
```

# 0/1 Knapsack — Dynamic Algorithm, II

```
27                  while ((k ≤ h) and (pair[k].p ≤ pair[next − 1].p)) do k := k + 1 ;
28              }
29              while (k ≤ h) do { // merge remaining terms from S_1^i
30                  pair[next].p := pair[k].p ; pair[next].w := pair[k].w ;
31                  next := next + 1 ; k := k + 1 ;
32              }
33              t := h + 1 ; h := next − 1 ; b[i + 1] := next ; // initialize for S_0^{i+1}
34          }
35          TraceBack(n, p, w, m, pair, x) ; // find solution x
36  }
```

- In the above algorithm
  - $pair$ is an array to store all feasible solutions, $S_0^i, 0 \le i \le n$.
  - $b$ is an array to store the indices of $S_0^i$ in $pair$ array
  - Function Largest$(pair, t, h, w[i], m)$ finds the largest $u$ satisfying

$$pair[u].w + w[i] \le m, \qquad t \le u \le h$$

  - The **for** loop of lines 10–34 generates $S_0^i, 1 \le i \le n$.
  - First $S_0^{i-1}$ is copied into $S_0^i$
  - Then $S_1^{i-1}$ is generated and merged into $S_0^i$
  - Lines 19-26 remove dominated entries



# 0/1 Knapsack — Example

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, $(x_1, x_2, x_3)$, $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit, $P = \sum_{i=1}^{3} p_i x_i$ .
- After executing the algorithm DKnap, we have

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $pair[\,].p$ | 0 | 0 | 1 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 5 | 6 |
| $pair[\,].w$ | 0 | 0 | 2 | 0 | 2 | 3 | 5 | 0 | 2 | 3 | 4 | 6 |
| | ↑ $b[0]$ | ↑ $b[1]$ | | ↑ $b[2]$ | | | | ↑ $b[3]$ | | | | |

- Note that $(p, w) = (3, 5) \in S_0^3$ but not $S_0^4$ since it is dominated by $(5, 4)$.
- The last entry, $(pp, ww) = (6, 6)$, is the optimal solution.

# 0/1 Knapsack — Example

- To find if each object is placed into the sack or not, $x[i], 1 \le i \le n$.
- One starts from $i = n$ and trace back to 1.
  - The optimal solution is $(pp, ww)$,
  - If $(pp, ww) \in S_0^n$ then $x[n] = 0$
    - $(pp_{n-1}, ww_{n-1}) = (pp, ww)$.
  - Otherwise $x[n] = 1$,
    - $(pp_{n-1}, ww_{n-1}) = (pp - p[n], ww - w[n])$.
- Repeat checking for $S_0^{n-i}$ and update $(pp_{n-i}, ww_{n-i})$, one finds the solution $x[i], 1 \le i \le n$.
- For the last example,
  - $(6, 6) \notin S^2$, thus $x[3] = 1$,
  - $(1, 2) \in S^1$, and $x[2] = 0$,
  - $(1, 2) \notin S^0$, thus $x[1] = 1$.
  - Optimal solution $x = (1, 0, 1)$, $(p, w) = (6, 6)$.

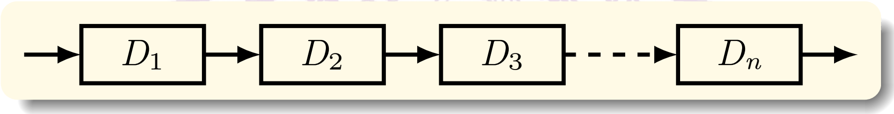


# 0/1 Knapsack — Complexity

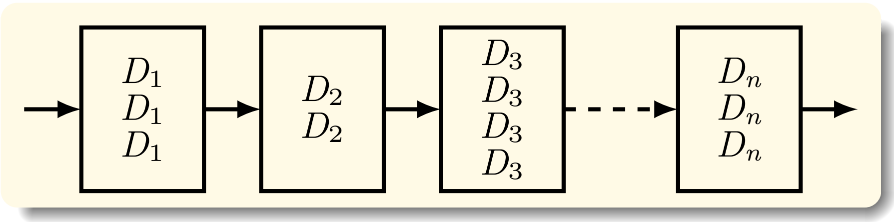
- Let the space needed to store $S_0^i$ in *pair* be $|S_0^i|$, then
$$|S_0^i| \le 2^{i-1}$$
And the total space needed for *pair* is
$$\sum_{i=1}^{n} |S_0^i| \le \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$
- Thus the space complexity is $\mathcal{O}(2^n)$
- The time needed to generate $S_0^i$ is $\Theta(S_0^{i-1})$, therefore the total time to generate all pairs is
$$\sum_{i=1}^{n} |S_0^{i-1}| \le \sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1} - 1$$
and the time complexity is $\mathcal{O}(2^n)$.
- The time complexity of the `Traceback` function is $\mathcal{O}(n^2)$ since it involves $n$ searches in the range $b[i]$ and $b[i+1]$.
  - Each search can take $\log(|S_0^i|) = \log(2^{i-1}) = (i - 1)\log 2$.
  - Total time is $\sum_{i=1}^{n} (i - 1)\log 2 = \mathcal{O}(n^2)$.

# System Reliability

- Suppose a system is composed of $n$ stages of devices connected in series.
  - Let $r_i$ be the reliability of device $D_i$ – the probability that device $D_i$ function normally.
  - Then the reliability of the system is $\prod_{i=1}^{n} r_i = r_1 r_2 \cdots r_n$.




- To improve the reliability of the system, one can replace stage $i$ by multiple, $m_i$, devices connected in parallel.
  - Then the reliability of stage $i$ becomes $\phi_i(m_i) = 1 - (1 - r_i)^{m_i}$.
  - The system reliability becomes $\prod_{i=1}^{n} \phi_i(m_i)$.






# Reliability Design Problem

- Assuming device $D_i$ costs $c_i$ each piece, and the total cost of the entire system is $c$, the reliability design problem is to find the multiplicity of each device, $m_i$ for each $D_i$ such that

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{n} \phi_i(m_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} c_i m_i \le c \\ \text{and} \quad & m_i \in N \text{ and } m_i \ge 1, \quad 1 \le i \le n. \end{aligned} \tag{6.3.3}$$

- Since $m_i \ge 1$ and $\sum c_i = c$, we can define

$$u_i = \lfloor (c + c_i - \sum_{j=1}^{n} c_j)/c_i \rfloor \tag{6.3.4}$$

- And the reliability design problem can be reformulated as

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{n} \phi_i(m_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} c_i m_i \le c \\ \text{and} \quad & 1 \le m_i \le u_i. \end{aligned} \tag{6.3.5}$$

# Reliability Design Problem, II

- Given the $n$ stages and the total cost of the optimal solution is $f_n(c)$, then the multiplicity, $m_n$, for stage $n$ should be determined by

$$f_n(c) = \max_{m_n=1}^{u_n} \phi_n(m_n) f_{n-1}(c - c_n m_n) \qquad (6.3.6)$$

  It is also assumed that $f_0(c) = 1$ for any $c$.
- Then this problem is similar to the 0/1 knapsack problem and the dynamic approach can be used to find the solution of the problem.
- Example, 3 devices, $D_1$, $D_2$ and $D_3$, with $r_1 = 0.9$, $r_2 = 0.8$ $r_3 = 0.5$, $c_1 = 30$, $c_2 = 15$, $c_3 = 20$, and the total cost $c \leq 105$. (It can derived that $u_1 = 2$, $u_2 = 3$ and $u_3 = 3$).



# Summary

- String editing problem
  - $\mathcal{O}(mn)$
- 0/1 knapsack problem
  - $\mathcal{O}(2^n)$
- System reliability design
  - Large time complexity