Unit 5.1 The Greedy Method

Algorithms

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Knapsack Problem

- Knapsack problem
  - Given n objects, each object i, $1 \leq i \leq n$, has
    - Weight w_i,
    - Profit $p_i \cdot x_i$, if x_i fraction is placed into the bag ($0 \leq x_i \leq 1$).
  - A bag with capacity m.
  - The objective is to maximize the profit.

$$\text{maximize} \quad \sum_{i=1}^{n} p_i x_i, \tag{5.1.1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} w_i x_i \leq m, \tag{5.1.2}$$

$$\text{and} \quad 0 \leq x_i \leq 1, \qquad 1 \leq i \leq n. \tag{5.1.3}$$

- A feasible solution is any set $\{x_1, \cdots, x_n\}$ that satisfies Eqs. (5.1.2) and (5.1.3).
- An optimal solution is a feasible solution for which Eq. (5.1.1) is maximized.

Knapsack Problem – Example

- An example of knapsack problem
  - $n = 3$, $m = 20$, $\{p_1, p_2, p_3\} = \{25, 24, 15\}$, and $\{w_1, w_2, w_3\} = \{18, 15, 10\}$.
  - Four feasible solutions

Solution	$\{x_1, x_2, x_3\}$	$\sum w_i x_i$	$\sum p_i x_i$
1	$\{1/2, 1/3, 1/4\}$	16.5	24.25
2	$\{1, 2/15, 0\}$	20	28.2
3	$\{0, 2/3, 1\}$	20	31
4	$\{0, 1, 1/2\}$	20	31.5

- Note that $\sum w_i x_i \le m$ for all 4 feasible solutions.
- Solution 4 yields the maximum profit among these 4 feasible solutions.

Knapsack Problem – Properties

Lemma 5.1.1.

In case the sum of all the weights is less than or equal to m, i.e., $\sum_{i=1}^{n} w_i \le m$, then $x_i = 1$, $1 \le i \le n$, is an optimal solution.

Lemma 5.1.2.

In case $\sum_{i=1}^{n} w_i \ge m$, then all optimal solutions will fill the knapsack exactly, i.e., $\sum_{i=1}^{n} w_i x_i = m$.

Knapsack Problem – Algorithm 1

- A general greedy algorithm for knapsack program is shown below.

Algorithm 5.1.3. Knapsack by Profit

1 **Algorithm** Knapsack_P(m, n, w, p, x)
2 // n objects with weight, $w[1:n]$, and profit, $p[1:n]$, find $x[1:n]$ that
3 // maximizes $\sum p[i]x[i]$ with $\sum w[i]x[i] \le m$, and $0 \le x[i] \le 1$.
4 {
5 $\quad A[1:n] :=$ Objects sorted by decreasing $p[1:n]$; // $p[A[i]] \ge p[A[j]]$ if $i < j$.
6 $\quad$ **for** $i := 1$ **to** n **do** $x[i] := 0$;
7 $\quad i := 1$;
8 $\quad$ **while** $(i \le n$ **and** $w[A[i]] \le m)$ **do** {
9 $\qquad x[A[i]] := 1$; $m := m - w[A[i]]$; $i := i + 1$;
10 $\quad$ }
11 $\quad$ **if** $(i \le n)$ **then** $x[A[i]] := m/w[A[i]]$;
12 }

- Note that line 4 sort A into *decreasing order* by p
- Applying this algorithm we get solution 2 for the example.

Knapsack Problem – Algorithm 2

- The greedy algorithm can be modified as below.

Algorithm 5.1.4. Knapsack by Weight

1 **Algorithm** Knapsack_W(m, n, w, p, x)
2 // n objects with weight, $w[1:n]$, and profit, $p[1:n]$, find $x[1:n]$ that
3 // maximizes $\sum p[i]x[i]$ with $\sum w[i]x[i] \le m$, and $0 \le x[i] \le 1$.
4 {
5 $\quad A[1:n] :=$ Objects sorted by increasing $w[1:n]$; // $w[A[i]] \le w[A[j]]$ if $i < j$.
6 $\quad$ **for** $i := 1$ **to** n **do** $x[i] := 0$;
7 $\quad i := 1$;
8 $\quad$ **while** $(i \le n$ **and** $w[A[i]] \le m)$ **do** {
9 $\qquad x[A[i]] := 1$; $m := m - w[A[i]]$; $i := i + 1$;
10 $\quad$ }
11 $\quad$ **if** $(i \le n)$ **then** $x[A[i]] := m/w[A[i]]$;
12 }

- Note that line 4 sort A into *increasing order* by w
- Applying this algorithm we get solution 3 for the example.

Knapsack Problem – Algorithm 3

- Another version of greedy algorithm is shown below.

Algorithm 5.1.5. Knapsack

```
1 Algorithm Knapsack(m, n, w, p, x)
2 // n objects with weight, w[1 : n], and profit, p[1 : n], find x[1 : n] that
3 // maximizes ∑ p[i]x[i] with ∑ w[i]x[i] ≤ m, and 0 ≤ x[i] ≤ 1.
4 {
5       A[1 : n] := Objects sorted by decreasing p[1 : n]/w[1 : n] ;
6             // p[A[i]]/w[A[i]] ≥ p[A[j]]/w[A[j]] if i < j.
7       for i := 1 to n do x[i] := 0 ;
8       i := 1 ;
9       while (i ≤ n and w[A[i]] ≤ m) do {
10            x[A[i]] := 1 ; m := m − w[A[i]] ; i := i + 1 ;
11      }
12      if (i ≤ n) then x[A[i]] := m/w[A[i]] ;
13 }
```

- Note that line 4 sort A into *decreasing order* by p/w
- Applying this algorithm we get solution 4 for the example.
  - This is the optimal solution since p/w is the real objective.

Knapsack Problem – Complexity and Optimality

- Knapsack Algorithm (Algorithm 5.1.5) has the time complexity of $\mathcal{O}(n \lg n)$.
  - Dominated by the Sort function on line 4
  - The while loop (lines 7-11) and for (line 5) loop are both $\mathcal{O}(n)$.

Lemma 5.1.6.

In case that the capacity is smaller than the weight of any object, $m < w_i$, $\forall i$, then the optimal solution is $x_i = m/w_i$, where p_i is the maximum, and $x_j = 0$, $j \neq i$.

Theorem 5.1.7.

If A is sorted by $\{p_i/w_i\}$ in non-increasing order, then the Knapsack algorithm (Algorithm 5.1.5) generates an optimal solution to the instance of the knapsack problem.

- Proof please see textbook [Horowitz], pp. 221-222.
- From Lemma (5.1.6), to fill a unit capacity the object with the maximum profit is the best choice, thus, the order should should be selected by p_i/w_i.

Container Loading

- Container loading problems
  - Input: n containers with w_i, $1 \le i \le n$, weight each.
  - A ship with cargo capacity of c.
  - Load the maximum number of containers to the ship.
- Let $x_i \in \{0, 1\}$ such that $x_i = 1$ if container i is loaded onto the ship.
  - The constraint is

$$\sum_{i=1}^{n} x_i w_i \le c. \tag{5.1.4}$$

  - The objective function to be maximized is

$$\sum_{i=1}^{n} x_i. \tag{5.1.5}$$

- Example: Suppose there are 8 containers with weights $[w_1, w_2, \cdots w_8] = [100, 200, 50, 90, 150, 50, 20, 80]$ and ship capacity $c = 400$.
  - Then the solution is $[x_1, x_2, \cdots, x_8] = [1, 0, 1, 1, 0, 1, 1, 1]$.
  - $\sum_{i=1}^{8} w_i x_i = 390$ that satisfies the constraint.
  - $\sum_{i=1}^{8} x_i = 6$ is the maximum number of containers loaded.

Container Loading – Algorithm

Algorithm 5.1.8. Container Loading

```
Algorithm ContainerLoading(c, n, w, x)
// n containers with weights w[1 : n] find x[1 : n] that maximizes ∑ x_i.
{
    A[1 : n] := Containers sorted by increasing w[1 : n] ;
        // w[A[i]] ≤ w[A[j]] if i < j.
    for i := 1 to n do x[i] := 0 ;
    i := 1 ;
    while (i ≤ n and w[A[i]] ≤ c) do {
        x[A[i]] := 1 ; c := c − w[A[i]] ; i := i + 1 ;
    }
}
```

- Note that $w[A[i]]$ is sorted into non-decreasing order.
  - Using the last example, $w[1 : 8] = \{100, 200, 50, 90, 150, 50, 20, 80\}$, then $A[1 : 8] = \{7, 3, 6, 8, 4, 1, 5, 2\}$ such that $w[A[i]]$ is in non-decreasing order.

Container Loading – Complexity and Optimality

- The time complexity of the `ContainerLoading` algorithm is dominated by the `Sort` function (line 4), which is $\mathcal{O}(n \lg n)$.
- The `while` loop (lines 7-10) is $\mathcal{O}(n)$.
- Overall complexity $\mathcal{O}(n \lg n)$.

Theorem 5.1.9.

The Container Loading Algorithm (Algorithm 5.1.8) generates optimal loading.

- Proof see textbook [Horowitz], pp. 215-217.

- Note that selecting the object with the least weight maximizes the capacity of loading the remaining objects.

Optimization Problems

- A special class of problems that has n inputs,
  - Arrange the inputs to satisfy some constraints – feasible solutions
  - Find feasible solution that minimize or maximize an objective function – optimal solution
- The greedy method is a algorithm that takes one input at a time
  - If a particular input results in infeasible solution, then it is rejected; otherwise it is included.
  - The input is selected according to some measure
  - The selection measure can be the objective functions or other functions that approximate the optimality
  - However, this method usually generates a suboptimal solution.

Greedy Method

- The following is an abstraction of the greedy method in subset paradigm

Algorithm 5.1.10. Greedy Method

```
Algorithm Greedy(A, n)
// A[1 : n] contains the n inputs.
{
    solution := ∅ ;
    for i := 1 to n do {
        x := Select(A) ; A := A − {x} ;
        if Feasible(solution ∪ x) then
            solution := solution ∪ x;
    }
    return solution;
}
```

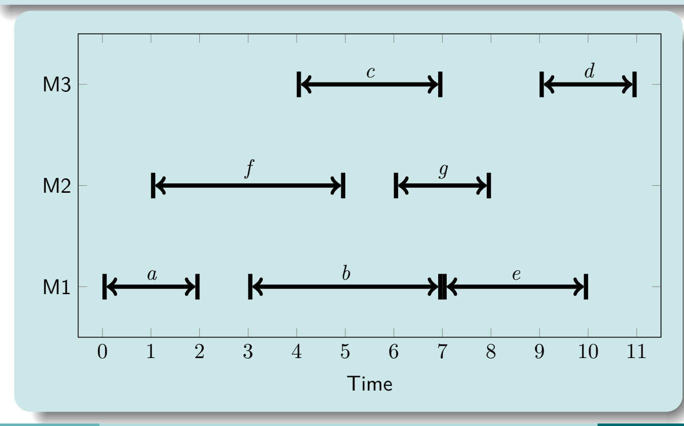
- In this subset paradigm the Select function selects an input from A and removes it.
- The Feasible function determines if it can be included into the solution vector.

- A variation of the greedy method is the ordering paradigm.
  - The inputs are ordered first and thus the Select function is not needed.

Machine Scheduling Problem

- Machine schedule problem
  - Input: n tasks and infinite number of machines
  - Each task has a start time $s[1 : n]$ and finish time, $f[1 : n]$, $s[i] < f[i]$.
  - Two tasks i and j overlap if and only if their processing intervals overlap at a point other than the interval start or end times.
  - A feasible task-to-machine assignment is that no machine is assigned with overlapping tasks.
  - An optimal assignment is a feasible assignment that utilizes the fewest number of machines.
- Example

Task	a	b	c	d	e	f	g
Start time	0	3	4	9	7	1	6
Finish time	2	7	7	11	10	5	8



Machine Scheduling Problem – Algorithm

Algorithm 5.1.11. Machine Scheduling

```
1  Algorithm MachineSchedule(tasks, n, s, t, M)
2  // Assign n tasks with start and times, s[1 : n], t[1 : n], to m machines.
3  // m is minimum and M[1 : n] is the assignment.
4  {
5        A[1 : n] := tasks sorted by increasing s[1 : n] ;
6            // s[A[i]] ≤ s[A[j]], if i < j.
7        m := 1 ; M[A[1]] := m ;
8        for i := 2 to n do {
9            j := {j | f[A[j]] = min_{1≤k≤i} f[A[k]]} ;
10               // Minimum finish time among all scheduled tasks.
11           if (f[A[j]] ≤ s[A[i]]) then // Machine processing A[j] is available
12               M[A[i]] := M[A[j]]) ; // Assign task A[i] to machine M[A[j]]
13           else {
14               m := m + 1 ; // Need more more machines
15               M[A[i]] := m ; // Assign task A[i] to machine m.
16           }
17       }
18 }
```


Machine Scheduling Problem – Complexity

Theorem 5.1.12.

The Machine Scheduling Algorithm (Algorithm 5.1.11) generates an optimal assignment.

- In Algorithm (5.1.11), the time complexity is dominated by
  - Sort function on line 4: $\mathcal{O}(n \lg n)$
  - Min function on line 7: $\mathcal{O}(\lg n)$
    - In a for loop and thus $\mathcal{O}(n \lg n)$
  - Total complexity: $\mathcal{O}(n \lg n)$.

Summary

- Knapsack problem
- Container loading problem
- Greedy method
- Machine scheduling problem