Unit 3.3 More on Divide and Conquer Algorithms

Algorithms

EE/NTHU

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Selection Algorithm

- The selection problem is to find the k'th element of an array A and place all elements less than or equal to $A[k]$ in $A[1:k-1]$ and the rest in $A[k+1:n]$.
- Divide and conquer can be applied to this problem as well.
  - The `Partition` algorithm can be effective in this selection problem.

Algorithm 3.3.1. Selection

```
Algorithm Select1(A, n, k)
// Partition the array into A[1 : k−1] ≤ A[k] ≤ A[k+1 : n].
{
      low := 1 ; high := n+1 ; A[n+1] := ∞ ; j := k−1 ;
      while (j ≠ k) do {
            j := Partition(A, low, high) ;
            if (k < j) then high := j;
            else if (k > j) then low := j+1 ;
      }
}
```

- After completing `Select1`, $A[k]$ is the k'th element.

Selection Algorithm – Complexity

- Note that the Partition$(A, low, high)$ decrease the range of the array A by at least 1.
- Thus, the worst-case complexity of the Select1 algorithm is $\mathcal{O}(n^2)$.
- Let $T_A^k(n)$ be the average time to find the k'th smallest element in $A[1:n]$
  - The average is taken over all $n!$ different permutations.
- Define

$$T_A(n) = \frac{1}{n}\sum_{k=1}^{n} T_A^k(n) \tag{3.3.1}$$

$$R(n) = \max_k T_A^k(n) \tag{3.3.2}$$

  - $T_A(n)$ is the average execution time of Select1 algorithm,
  - And it is obvious that $T_A(n) \le R(n)$.

Selection Algorithm – Complexity, II

Theorem 3.3.2.

The average execution time $T_A(n)$ of Select1 algorithm is $\mathcal{O}(n)$.

Proof. The complexity of Partition algorithm is $\mathcal{O}(n)$ and hence there is a constant c such that

$$T_A^k(n) \le c \cdot n + \frac{1}{n}\Big(\sum_{i=1}^{k-1} T_A^k(n-i) + \sum_{i=k+1}^{n} T_A^k(i-1)\Big),$$

$$R(n) \le c \cdot n + \frac{1}{n}\max_k \Big(\sum_{i=1}^{k-1} R(n-i) + \sum_{i=k+1}^{n} R(i-1)\Big),$$

$$R(n) \le c \cdot n + \frac{1}{n}\max_k \Big(\sum_{i=n-k+1}^{n-1} R(i) + \sum_{i=k}^{n-1} R(i)\Big).$$

To show by induction that $R(n) \le 4c \cdot n$.

Selection Algorithm – Complexity, III

For $n = 2$,

$$R(n) \leq 2 \cdot c + \frac{1}{2} \max \Big(R(1), R(1) \Big)$$
$$\leq 2.5 \cdot c < 4 \cdot c \cdot n$$

Next assume $R(n) \leq 4 \cdot c \cdot n$ for $2 \leq n < m$.
For $n = m$,

$$R(m) \leq c \cdot m + \frac{1}{m} \max_k \Big(\sum_{i=m-k+1}^{m-1} R(i) + \sum_{i=k}^{m-1} R(i) \Big).$$

Since $R(n)$ is a nondecreasing function of n, $\sum_{i=m-k+1}^{m-1} R(i) + \sum_{i=k}^{m-1} R(i)$ is maximum when $k = m/2$ when m is even, and $k = (m+1)/2$ when m is odd.
When m is even

$$R(m) \leq c \cdot m + \frac{2}{m} \sum_{i=m/2}^{m-1} R(i)$$
$$\leq c \cdot m + \frac{8c}{m} \sum_{i=m/2}^{m-1} i$$
$$\leq 4 \cdot c \cdot m$$

Selection Algorithm – Complexity, IV

When m is odd

$$R(m) \leq c \cdot m + \frac{2}{m} \sum_{i=(m+1)/2}^{m-1} R(i)$$
$$\leq c \cdot m + \frac{8c}{m} \sum_{i=(m+1)/2}^{m-1} i$$
$$\leq 4 \cdot c \cdot m$$

Since $T_A(n) \leq R(n)$, therefore $T_A(n) \leq 4 \cdot c \cdot n$ and $T_A(n)$ is $\mathcal{O}(n)$. $\square$

- The space complexity of the `Select1` algorithm is $\mathcal{O}(n)$ for the array A.
- The `Select1` algorithm can also be randomized as `RQuickSort` algorithm.
  - The expected time complexity is still $\mathcal{O}(n)$.
  - But the average performance is expected to be better.

Selection Algorithm – Complexity, V

- The execution time of Select1 in the worst-case is $\mathcal{O}(n^2)$.
  - The worst-case can happen if the partition element, $A[low]$, is close to extreme.
  - If the partition element is close to the median, $A[(low + high)/2]$, then the number of iterations can be reduced significantly.
  - Using this argument, the selection algorithm is modified to have worst-case linear time complexity.
- The array A is divided into subarrays each has r elements
  - $\lceil n/r \rceil$ groups
  - A small r is usually prefered ($r = 5$, for example).
- Then the median of each group is found and move to the front array A
- The median of the medians, mm, is then found using Partition function
- Now, this mm can be used to partition array A.
- Since mm is used for each partition step in the selection algorithm, worst-case linear time can be guaranteed.
- Note that though Select2 is worst-case linear, it has a much larger coefficient, as compared to Select1, thus for small to median-size problems, Select2 may not be faster in execution.
  - Select2 returns the position j such that $A[j]$ is the k'th element.

Selection Algorithm – Worst-case Linear

Algorithm 3.3.3. Selection – Worst-case Linear

```
Algorithm Select2(A, k, low, high, r)
// Partition the array into A[1 : k−1] ≤ A[k] ≤ A[k+1 : n].
{
    j := k + low − 1;
    while (k ≠ j − low) do {
        n := high − low + 1;
        if (n ≤ r) then { // small array
            InsertionSort(A, low, high);
            return low + k;
        }
        for i := 1 to ⌈n/r⌉ do { // find median of each group and move to front
            InsertionSort(A, low + (i − 1) ∗ r, low + i ∗ r − 1);
            Swap(low + i − 1, low + (i − 1) ∗ r + ⌈r/2⌉ −1);
        }
        j := Select2(A, ⌈⌊n/r⌋/2⌉, low, low + ⌈n/r⌉ −1); // find median of medians
        Swap(low, j); // move median of median to A[low]
        j := Partition(A, low, high + 1);
        if (k < j − low) high := j; // reduce to A[low : j]
        else if (k > j − low) { // reduce to A[j + 1 : high]
            k := k − (j − low + 1);
            low := j + 1;
        }
    }
    return j;
}
```

Matrix Multiplication

- Given two $n \times n$ matrix $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}[i,j] \in \mathbb{R}$, $\mathbf{B}[i,j] \in \mathbb{R}$, $1 \le i, j, \le n$, then $n \times n$ matrix $\mathbf{C}$ is the product of $\mathbf{A}$ and $\mathbf{B}$, ($\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$),

$$\mathbf{C}[i,j] = \sum_{k=1}^{n} \mathbf{A}[i,k] \times \mathbf{B}[k,j], \qquad 1 \le i, j \le n. \tag{3.3.3}$$

- Note that to calculate $\mathbf{C}[i,j]$, one needs n multiplications and $n-1$ additions.
- Thus to calculate $\mathbf{C}$, which has n^2 elements, the time complexity is $\Theta(n^3)$.

Matrix Multiplication – Divide and Conquer

- Suppose $n = 2^k$, we can apply divide and conquer approach to matrix multiplication problem.
- Divide each matrix into 4 submatrices with $\frac{n}{2} \times \frac{n}{2}$ dimensions each, then

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{3.3.4}$$

where

$$\begin{aligned} \mathbf{C}_{11} &= \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{C}_{12} &= \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{C}_{21} &= \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{C}_{22} &= \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{aligned} \tag{3.3.5}$$

- To calculate matrix $\mathbf{C}$, we need
  - Eight matrix multiplications ($\frac{n}{2} \times \frac{n}{2}$),
  - Four matrix additions ($\mathcal{O}(n^2)$ complexity due to n^2 elements in $\mathbf{C}$).
- Let $T(n)$ be the complexity, then

$$T(n) = \begin{cases} b, & n \le 2 \\ 8 \cdot T(n/2) + c \cdot n^2, & n > 2. \end{cases} \tag{3.3.6}$$

where b and c are constants.

Matrix Multiplication – Divide and Conquer, II

- If $n = 2^k$

$$
\begin{aligned}
T(n) &= 8\,T(n/2) + c \cdot n^2 \\
&= 8\left[8\,T(n/4) + c \cdot \left(\frac{n}{2}\right)^2\right] + c \cdot n^2 \\
&= 8^2\,T(n/4) + c \cdot n^2\,(2+1) \\
&= 8^3\,T(n/8) + c \cdot n^2\,(4+2+1) \\
&= 8^{k-1}\,T(n/2^{k-1}) + c \cdot n^2 \sum_{i=0}^{k-2} 2^i \\
&= 2^{3k-3}\,b + c \cdot n^2\left(\cdot 2^{k-1}\right) \\
&= \frac{n^3}{8}\,b + c \cdot n^2\left(\frac{n}{2}\right) \\
&= \left(\frac{b}{8} + \frac{c}{2}\right) n^3 \\
&= \mathcal{O}(n^3)
\end{aligned}
$$

- Thus, this divide and conquer approach does not improve the computational complexity

Strassen's Matrix Multiplication

- Given Equations (3.3.4) and (3.3.5), define the following

$$
\begin{aligned}
\mathbf{P} &= (\mathbf{A}_{11} + \mathbf{A}_{22})(\mathbf{B}_{11} + \mathbf{B}_{22}) \\
\mathbf{Q} &= (\mathbf{A}_{21} + \mathbf{A}_{22})\mathbf{B}_{11} \\
\mathbf{R} &= \mathbf{A}_{11}(\mathbf{B}_{12} - \mathbf{B}_{22}) \\
\mathbf{S} &= \mathbf{A}_{22}(\mathbf{B}_{21} - \mathbf{B}_{11}) \\
\mathbf{T} &= (\mathbf{A}_{11} + \mathbf{A}_{12})\mathbf{B}_{22} \\
\mathbf{U} &= (\mathbf{A}_{21} - \mathbf{A}_{11})(\mathbf{B}_{11} + \mathbf{B}_{12}) \\
\mathbf{V} &= (\mathbf{A}_{12} - \mathbf{A}_{22})(\mathbf{B}_{21} + \mathbf{B}_{22})
\end{aligned}
\tag{3.3.7}
$$

  Then

$$
\begin{aligned}
\mathbf{C}_{11} &= \mathbf{P} + \mathbf{S} - \mathbf{T} + \mathbf{V} \\
\mathbf{C}_{12} &= \mathbf{R} + \mathbf{T} \\
\mathbf{C}_{21} &= \mathbf{Q} + \mathbf{S} \\
\mathbf{C}_{22} &= \mathbf{P} + \mathbf{R} - \mathbf{Q} + \mathbf{U}
\end{aligned}
\tag{3.3.8}
$$

- To find matrix $\mathbf{C}$, we need 7 matrix multiplications of $\frac{n}{2} \times \frac{n}{2}$ and 18 matrix additions.
- Since matrix multiplications, $\mathcal{O}(n^3)$, is more expensive than matrix addition, $\mathcal{O}(n^2)$, for large n this approach might be more efficient.

Strassen's Matrix Multiplication, II

- The recurrence relation for the computation time $T(n)$ is

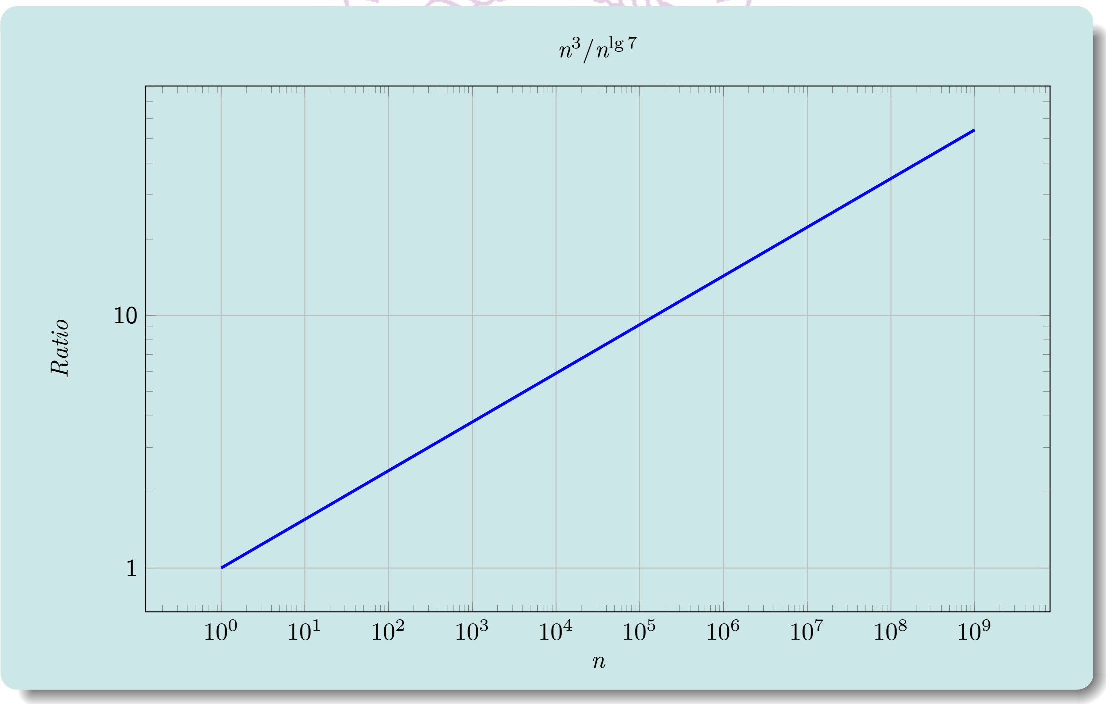
$$T(n) = \begin{cases} b, & n \le 2, \\ 7 \cdot T(n/2) + c \cdot n^2, & n > 2. \end{cases} \tag{3.3.9}$$

  where b and c are two constants.
- If $n = 2^k$, then

$$\begin{aligned} T(n) &= 7 \cdot T(n/2) + c \cdot n^2 \\ &= 7^2 \cdot T(n/4) + 7 \cdot c \cdot (n/2)^2 + c \cdot n^2 \\ &= 7^2 \cdot T(n/4) + c \cdot n^2 (7/4 + 1) \\ &= 7^{k-1} \cdot T(n/2^{k-1}) + c \cdot n^2 \sum_{i=0}^{k-2} (7/4)^i \\ &= 7^{k-1} \cdot b + c \cdot n^2 \left((7/4)^{k-1} - 1 \right) / (3/4) \\ &\approx n \frac{\lg 7}{7} + c' n^{\lg 4 + \lg 7 - \lg 4} \\ &= \mathcal{O}(n^{\lg 7}) = \mathcal{O}(n^{2.807}) \end{aligned}$$

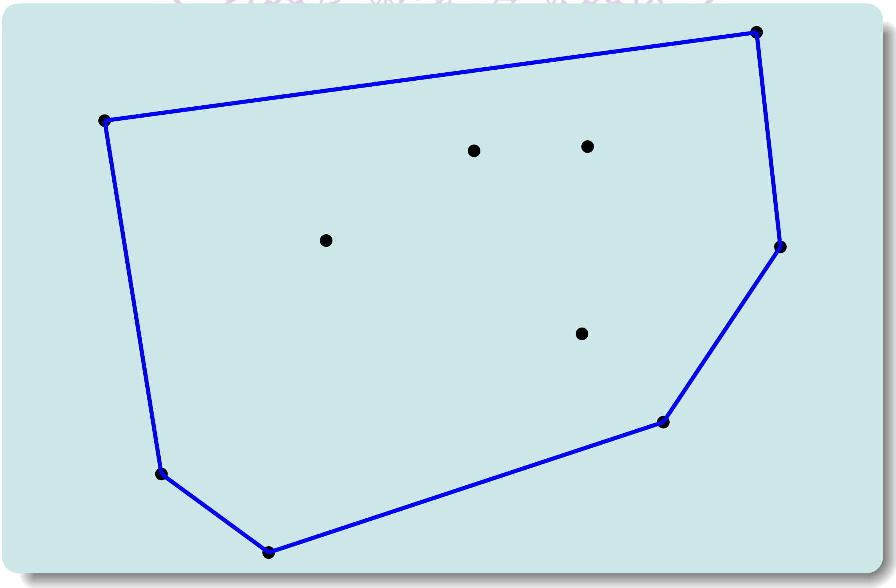
Strassen's Matrix Multiplication, III

- Compared to direct matrix multiplication, $\mathcal{O}(n^3)$, Strassen's approach can be faster for large n.
  - But, coding is much more complex



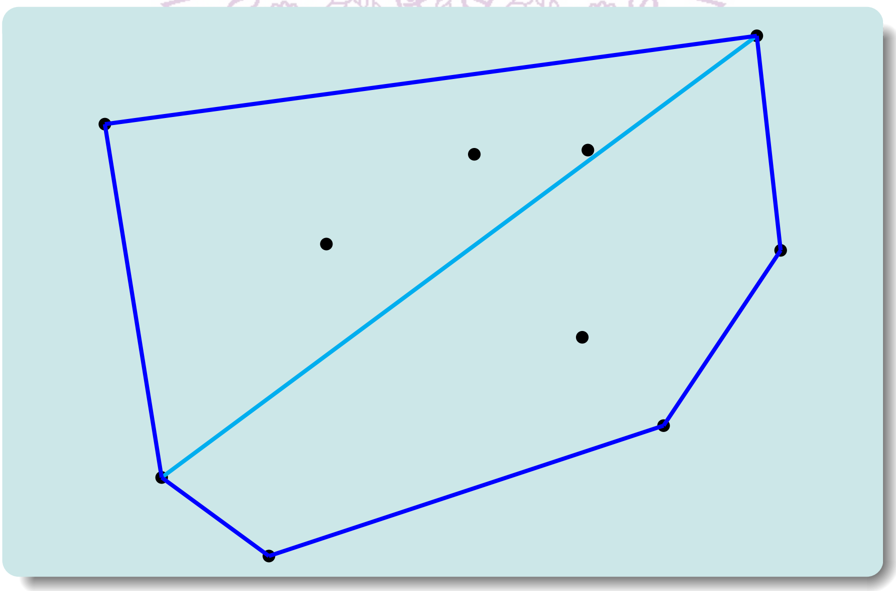
Convex Hull Problem

- Given a set S that contains points on a 2-D plane, the convex hull is defined as the smallest convex polygon that contains all the points in S.
- A polygon is convex if for any two points p_1, p_2 inside of of the polygon, the straight line segment connecting p_1 and p_2 is fully contained in the polygon.
- The vertices of the convex hull of a set S is a subset of S.
  - But, not necessarily a proper subset.



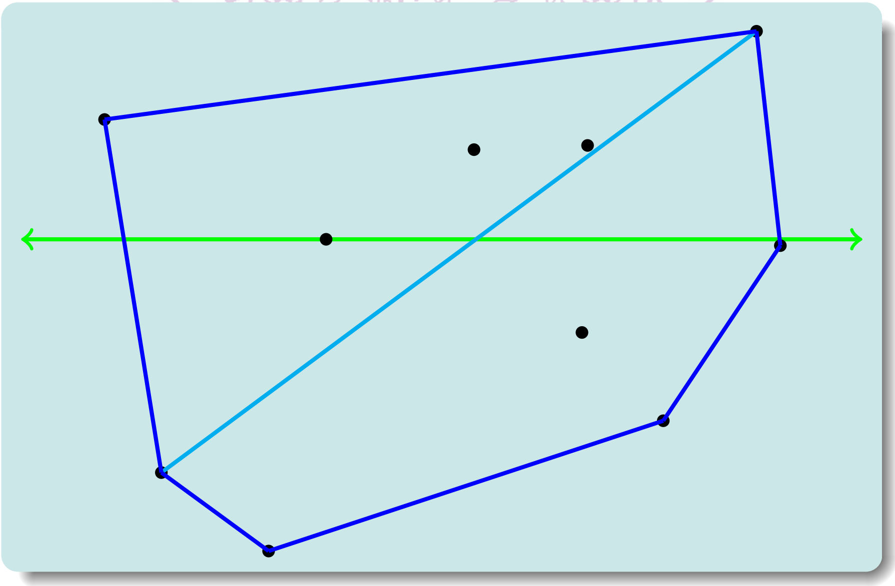
Convex Hull – Direct Implementation

- The convex hull of S can be found using the definition above
  - For any $p_1 \in S$, if p_1 is inside the triangle formed by $p_2, p_3, p_4 \in S$, with $p_1 \notin \{p_2, p_3, p_4\}$, then p_1 is not a vertex of the convex hull.
- This direct implementation has the time complexity of $\mathcal{O}(n^4)$.
  - n points to be tested, n^3 for all possible triangles.



Convex Hull – Direct Implementation, II

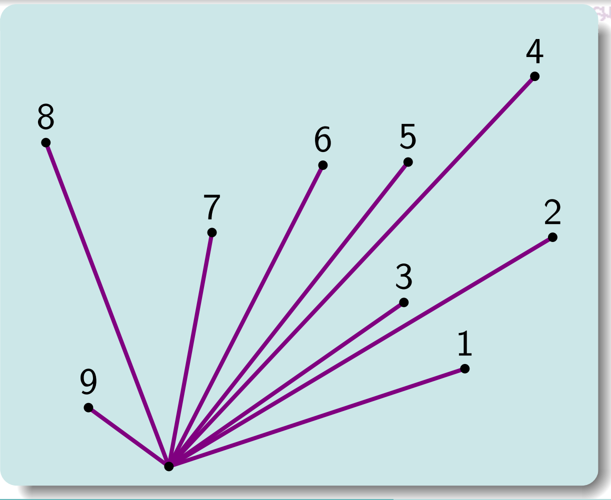
- To test if a point p_1 is inside of a triangle $\triangle p_2p_3p_4$
  - Let L be the horizontal line passing through $p_1 = (x_1, y_1)$, note that L can be described by the linear equation $y = y_1$, then check if L intersects with any of the line segments, $\overline{p_2p_3}$, $\overline{p_3p_4}$, $\overline{p_4p_2}$. If not, then p_1 is outside of $\triangle p_2p_3p_4$. Otherwise let (x_a, y_1) and (x_b, y_1) be the intersect points, if $x_a \le x_1 \le x_b$ then p_1 is inside of the triangle. (Note that it is possible $x_a = x_b$.)

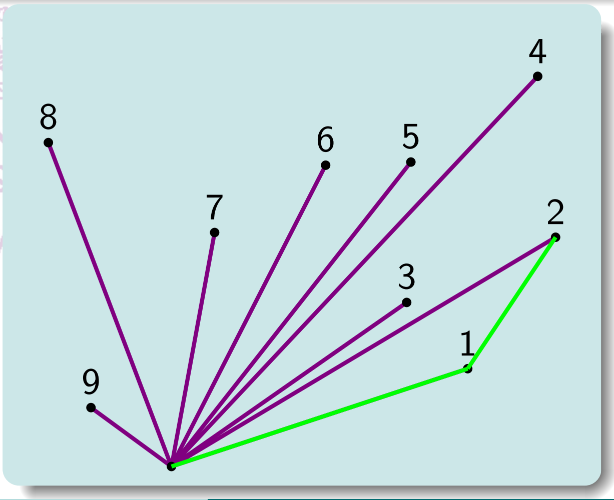


Convex Hull – Graham's Scan

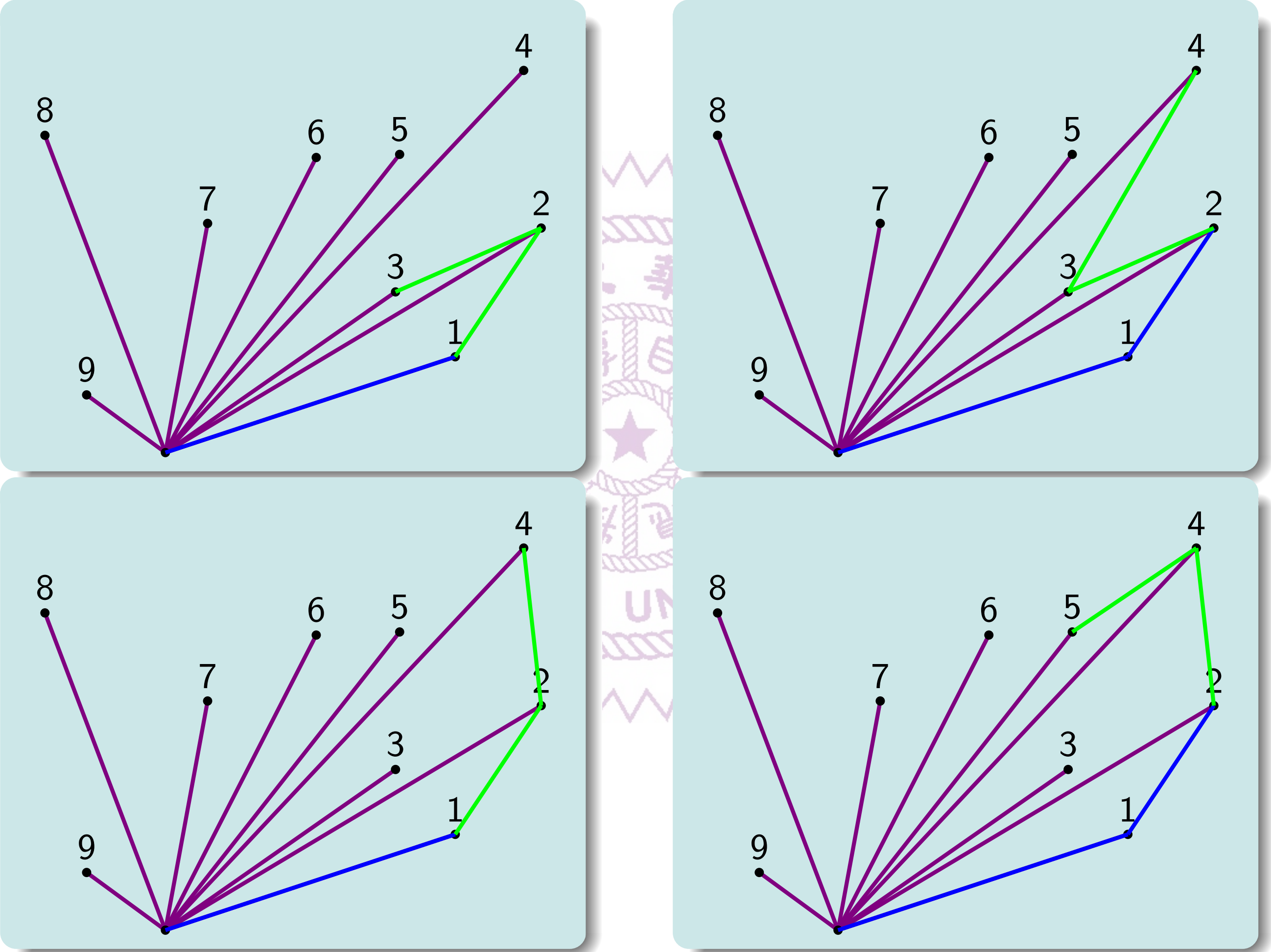
Algorithm 3.3.4. Convex Hull

```
Algorithm ConvexHull(ptslist)
// Find the convex hull in the ptrlist.
{
    Let the first element of ptrlist has the smallest y coordinate ;
    Sort(ptrlist) ; // Sort by angle between p and x-axis.
    Scan(ptrlist) ;
    PrintList(ptrlist) ;
}
```

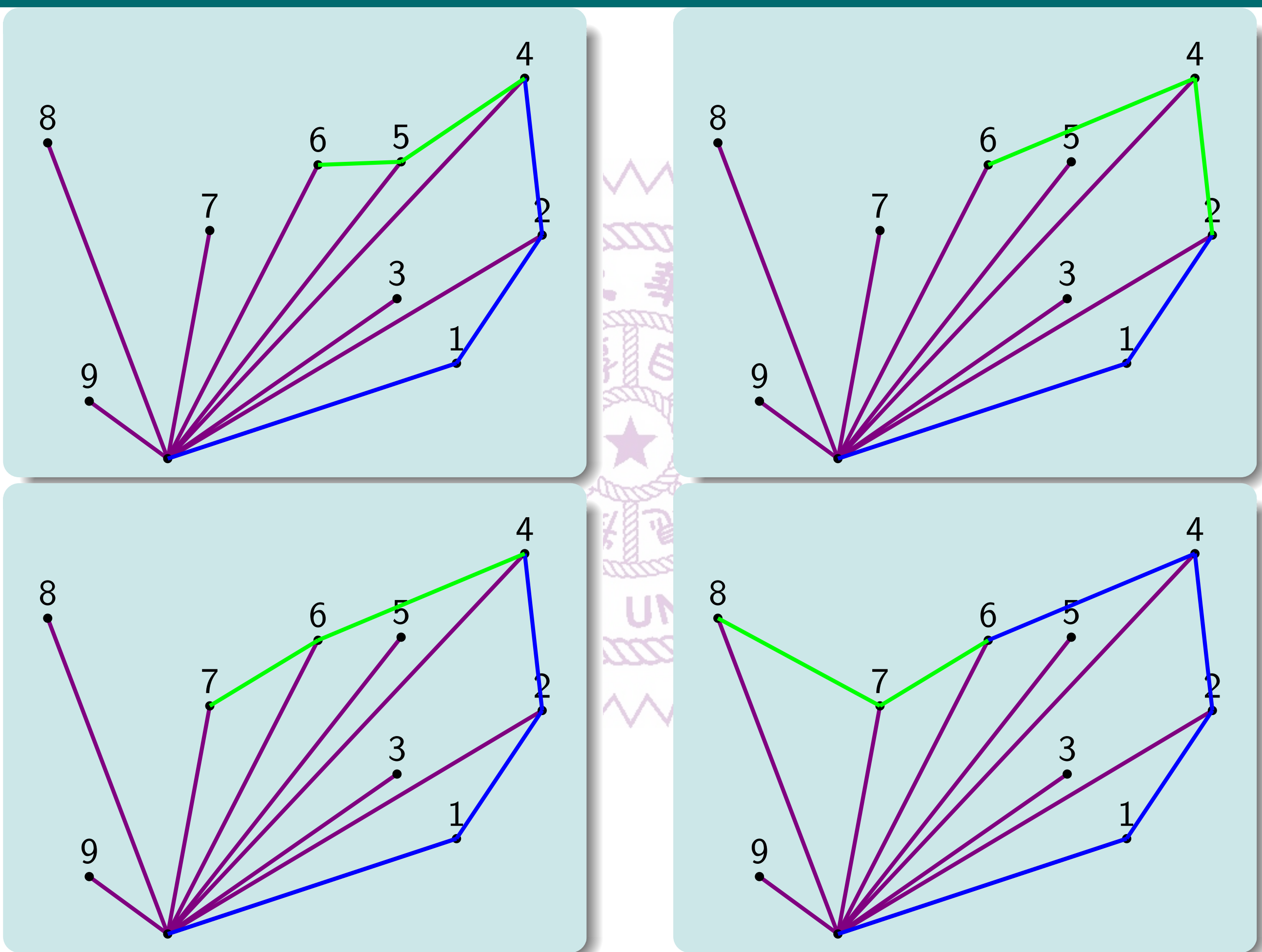




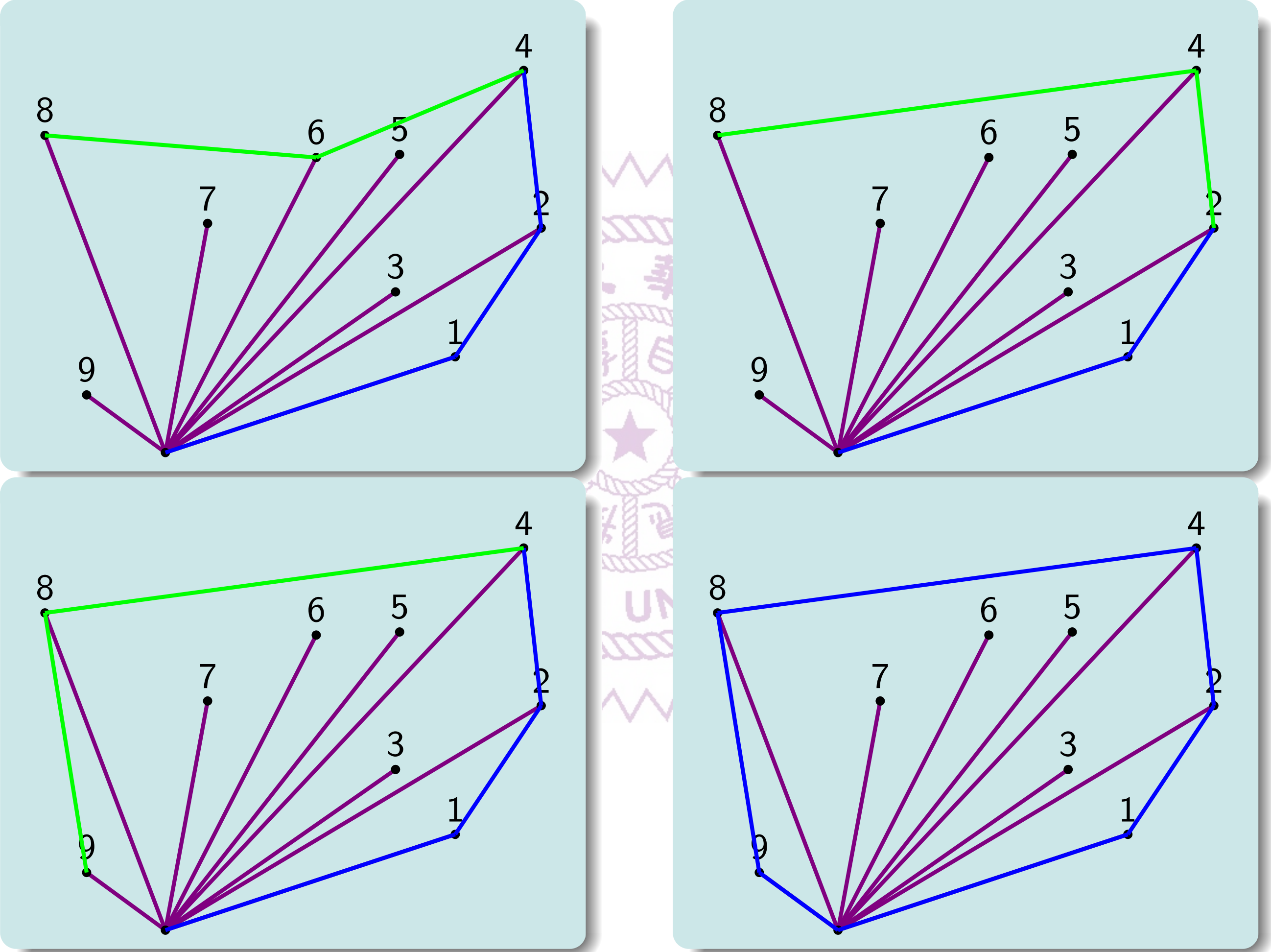
Convex Hull – Graham's Scan, II



Convex Hull – Graham's Scan, III



Convex Hull – Graham's Scan, IV



Convex Hull – Graham's Scan Algorithm

Algorithm 3.3.5. Graham's Scan Algorithm

```
Algorithm Scan(list)
// Remove internal points.
{
    p1 := list; p2 := p1 → next;
    while (p2 → next ≠ NULL) do { // For all points on the sorted list.
        p3 := p2 → next;
        if ( Area(p1, p2, p3) > 0) then // p1p2p3 turning left, accept p2.
            p1 := p1 → next;
        else {
            p1 → next := p3; // Remove p2 from the list.
            p3 → prev := p1;
            delete p2;
            p1 := p1 → prev; // Backtrack p1.
        }
        p2 := p1 → next;
    }
}
```

Convex Hull – Graham's Scan Algorithm, II

- In the preceding algorithm, the points are in linked list form consists

```
struct Point {
    double x,y;
    struct Point *next, *prev;
}
```

  - Note that this is a double linked list.
- Let $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ and $p_3(x_3, y_3)$ be three points in a plane the function `Area`(p_1, p_2, p_3) is defined as

$$\det \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{3.3.10}$$

- It can be shown that
  - If the area is positive then p_3 is located to the left of the vector $\overrightarrow{p_1 p_2}$.
  - If the area is negative then p_3 is located to the right of the vector $\overrightarrow{p_1 p_2}$.
  - If the area is zero then p_3 is colinear with $\overrightarrow{p_1 p_2}$.

Convex Hull – Graham's Scan Algorithm, Complexity

- The Algorithm (3.3.4) consists of 3 steps
  - (line 4) finding the first element with the smallest y coordinate can be done in $\mathcal{O}(n)$ time,
  - (line 5) sort by the angle can be done in $\mathcal{O}(n \lg n)$ time,
  - (line 6) Graham's Scan can be done in $\mathcal{O}(n)$ time.
- Thus, the time complexity is $\mathcal{O}(n \lg n)$.

Quick Hull Algorithm

- Divide and conquer approach can be used to find the convex hull.

Algorithm 3.3.6. QuickHull

Algorithm QuickHull(*list*, *CHull*)
// Generate Convex Hull for points in *list*.
{
    Find $p_1 \in list$ with the smallest x coordinate .
    Find $p_2 \in list$ with the largest x coordinate .
    Let $X_1 := \{p | \texttt{Area}(p_1, p_2, p) > 0\}$. // Upper half.
    Let $X_2 := \{p | \texttt{Area}(p_1, p_2, p) < 0\}$. // Lower half.
    Hull$(p_1, p_2, X_1, UpperHull)$; // Create upper hull.
    Hull$(p_2, p_1, X_2, LowerHull)$; // Lower hull.
    $CHull :=$ Merge$(UpperHull, LowerHull)$; // Merge them.
}

- Finding p_1 and p_2 takes $\mathcal{O}(n)$ time.
- Finding X_1 and X_2 takes $\mathcal{O}(n)$ time.
- Merge takes no more than $\mathcal{O}(n)$ time.
- The time complexity can be dominated by Hull function.

Quick Hull Algorithm, II

Algorithm 3.3.7. QuickHull

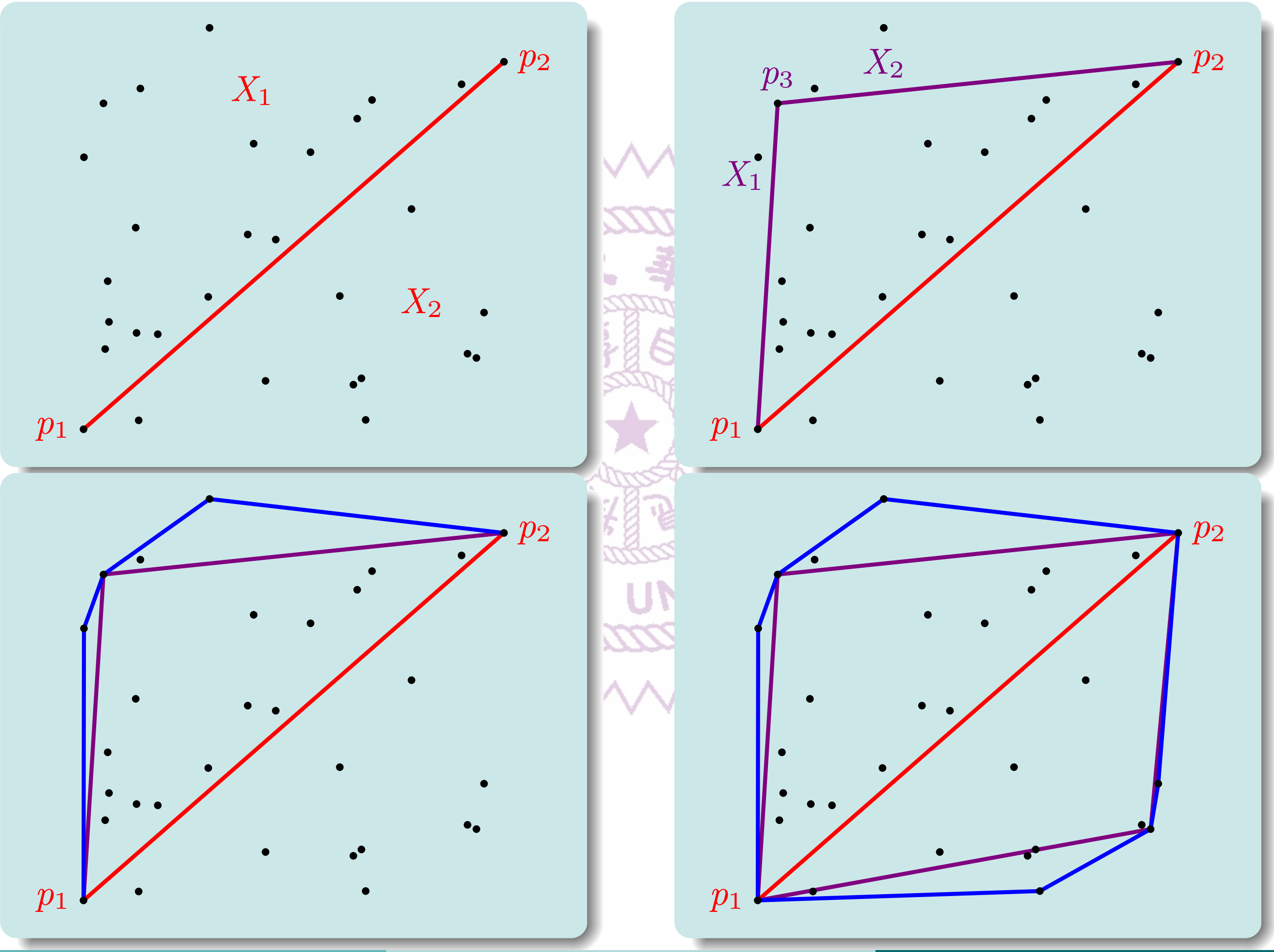
Algorithm Hull$(p_1, p_2, list, CHull)$
// Find convex *Hull* for p_1, p_2 and *list*.
{
    Find $p_3 \in list$ with the largest $|\texttt{Area}(p_1, p_2, p_3)|$;
    Let $X_1 := \{p | \texttt{Area}(p_1, p_3, p) > 0\}$. // All points left to $\overrightarrow{p_1 p_3}$/
    if $(X_1 = \emptyset)$ **then** $H_1 := \{p_1, p_3\}$; // No more points.
    else HULL(p_1, p_3, X_1, H_1) ; // Recursive call if more points.
    Let $X_2 := \{p | \texttt{Area}(p_3, p_2, p) > 0\}$.
    if $(X_2 = \emptyset)$ **then** $H_2 := \{p_3, p_2\}$;
    else HULL(p_3, p_2, X_2, H_2) ;
    $CHull :=$ Merge(H_1, H_2) ; // Combine those two hulls.
}

- Finding p_3, X_1, and X_2 take $\mathcal{O}(m)$ time, if *list* has m points.
- Thus, if $T(m)$ is the time for HULL algorithm we have

$$T(m) = T(m_1) + T(m_2) + \mathcal{O}(m), \quad (3.3.11)$$

  where $m_1 + m_2 \le m$.
- This recurrence relationship is the same as QuickSort.
  - Worst-case complexity is $\mathcal{O}(m^2)$, and average-case is $\mathcal{O}(m \lg m)$.

Quick Hull Example



Time Complexity of Divide and Conquer Algorithms

- In Algorithm DandC (Algorithm 3.1.1) the problem is divided into k subproblems; each solved recursively; then the results are combined to form the final solution.
- The execution time can be assumed to have a general recurrence equation as

$$T(n) = a \cdot T(n/k) + f(n). \tag{3.3.12}$$

  where $f(n)$ is the time to divide problem into k subsets and to combine the subsets to form the final solution.
- Let $n = k^m$, then

$$\begin{aligned}
T(n) &= a \cdot T(n/k) + f(n) \\
&= a \cdot \Big(a \cdot T(n/k^2) + f(n/k) \Big) + f(n) \\
&= a^2 \cdot T(n/k^2) + a \cdot f(n/k) + f(n) \\
&= a^m \cdot T(n/k^m) + \sum_{i=0}^{m-1} a^i \cdot f(n/k^i) \\
&= a^{\log_k n} \cdot T(1) + \sum_{i=0}^{m-1} a^i \cdot f(n/k^i) \\
&= n^{\log_k a} + \sum_{i=0}^{m-1} a^i \cdot f(n/k^i) \qquad (3.3.13)
\end{aligned}$$

Master Method

- Note that
  - $T(1)$ is taken out since it is a constant,
  - The summation of the second part has $m = \log_k n$ terms.

1. If there is a positive ϵ such that $n^{\log_k a} = n^\epsilon \cdot f(n)$ then for large n

$$T(n) = \Theta(n^{\log_k a}). \tag{3.3.14}$$

2. If there is a positive ϵ such that $n^{\log_k a} = f(n)/n^\epsilon$ and if $a \cdot f(n/k) \le c \cdot f(n)$ for some constant $c < 1$ for large n

$$T(n) = \Theta(f(n)). \tag{3.3.15}$$

3. If $f(n) = \Theta(n^{\log_k a})$ then

$$T(n) = \Theta(n^{\log_k a} \lg n). \tag{3.3.16}$$

   This comes from the m summation terms. $m = \log_k n = \Theta(\lg n)$.

- In general, the time complexity of the divide-and-conquer algorithms fall into one of the three scenarios as shown in Eqs. (3.3.14, 3.3.15, 3.3.16).
- However, exceptions exist.

Master Method – Example

- Example 1, Algorithm `MaxMin`

$$T(n) = 2\,T(n/2) + 2$$

  $a = 2,\ k = 2,\ n^{log_k a} = n$ and $f(n) = 2$.
  Use Eq. (3.3.14) we have $T(n) = \Theta(n)$.
- Example 2, Algorithm `MaxSubArray`

$$T(n) = 2\,T(n/2) + n$$

  $a = 2,\ k = 2,\ n^{log_k a} = n$ and $f(n) = n$.
  Use Eq. (3.3.16) we have $T(n) = \Theta(n \lg n)$.
- Example 3, `MatrixMultiplication`

$$T(n) = 8\,T(n/2) + n^2$$

  $a = 8,\ k = 2,\ n^{log_k a} = n^3$ and $f(n) = n^2$.
  Use Eq. (3.3.14) we have $T(n) = \Theta(n^3)$.

Master Method – Example

- Example 4,

$$T(n) = 2T(n/2) + n^2$$

$a = 2,\ k = 2,\ n^{log_k a} = n$ and $f(n) = n^2$.
Use Eq. (3.3.15) we have $T(n) = \Theta(n^2)$.
This can also be derived as follows.

$$\begin{aligned}
T(n) &= 2T(n/2) + n^2 \\
&= 2\Big(2T(n/4) + (n/2)^2\Big) + n^2 \\
&= 4T(n/4) + n^2(1 + 1/2) \\
&= 2^m T(n/2^m) + n^2 \sum_{i=0}^{m-1} 1/2^i \\
&= n + n^2 \cdot 2 \cdot (1 - 2^{-m}) \\
&= \Theta(n^2)
\end{aligned}$$

- Thus, the Master method can be effective to find the complexity of divide and conquer algorithms.

Summary

- Selection problem
- Matrix multiplication
  - Strassen's matrix multiplication
- Convex hull problem
  - Graham's scan algorithm
  - Quick hull algorithm
- Time complexity of divide and conquer algorithms
  - Master method