EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Midterm Exam

Coverage: Chapters 1–5

Date: April 13, 2023

Time: 13:20 – 15:10

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Notice:

1. *Write your name, student ID number on both question and answer sheet. At the end, you need to return both sheets.*
2. *No need to calculate the exact values.*
3. *There are 10 questions with total 105 points.*
4. *This is a* **closed book examination***.*
5. *You need to provide clear logical reasoning for your answer.*

Part I

The first two problems are multiple-choice problems, and there may be more than one correct answer. Each incorrect answer will result in a deduction of 2 points, and the score will be zero if three or more choices are incorrect.

Problem 1. (5 points) Let $A, B \subseteq S$ be the events in the sample space S. Which of the following statements are **False**?

(a) If $A \subseteq B$, then $P(A) \leq P(B)$.

(b) If $P(B) > 0$, then $P(A \mid B) \geq P(A)$.

(c) $P(A \cap B) \geq P(A) + P(B) - 1$.

(d) $P(A \cap B^c) = P(A \cup B) - P(B)$.

(e) $P(A) = P(A \cup B) - P(A^c \cap B)$.

Solution: (b)

(a) True.

(b) False. Given a counter example. Let $S = \{1, 2, 3\}$ be the sample space with equal probability for each element, $A = \{1\}$, and $B = \{2\}$. Then $P(A \mid B) = 0 \ngeq 1/3 = P(A)$.

(c) True. Let's rearrange the inequality into

$$1 \geq P(A) + P(B) - P(A \cap B) = P(A \cup B).$$

According to the axiom of the probability, the statement is true.

(d) True. It's easy to verify by Venn diagram.

(e) True. It's easy to verify by Venn diagram.

Problem 2. (5 points) Which of the following statements are **True**?

(a) If A, B are independent events, A and B^c are independent.

(b) If A, B are independent events, A^c and B are independent.

(c) If A, B are independent events, A^c and B^c are independent.

(d) For any random variable X, $E[1/X] = 1/E[X]$.

(e) Let X be Poisson (λ), and $a \in \mathbb{N}$. Then $P(X \geq a) \leq E[X]/a$.

Solution: (a)(b)(c)(e)

(a) True. Since A, B are independent, $P(A \cap B) = P(A)P(B)$. This implies

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c).$$

(b) True. Since A and B are arbitrary events, it is really the same statement as in part (a).

(c) True. To prove that A^c and B^c are independent, we apply the result of part (a).

$$P(A^c \cap B^c) = P(B^c) - P(A \cap B^c) = P(B^c) - P(A)P(B^c) = P(B^c)(1 - P(A)) = P(B^c)P(A^c).$$

(d) False. Given a counter example. Let $S = \{1, 2, 3\}$, and the probability $P(X = 1) = P(X = 2) = P(X = 3) = 1/3$. Then

$$E[1/X] = \frac{1}{1} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{11}{18} \neq \frac{1}{2} = 1/E[X].$$

(e) True. Since X is a non-negative random variable,

$$E[X] = \sum_{x=0}^{\infty} xP(X = x) \geq \sum_{x=a}^{\infty} xP(X = x) \geq \sum_{x=a}^{\infty} aP(X = x) = aP(X \geq a).$$

Thus, the statement is true.

Part II

The following questions are calculation problems, requiring a logical calculation process to be written, and the answer must be clearly marked.

Problem 3. (10 points) Calculate the probability $P(K < E[K])$ of the following distribution and parameters.

(a) (3 points) K is binomial with parameters $(n, p) = (6, 1/2)$.

(b) (3 points) K is Poisson with parameter $\lambda = 3$.

(c) (4 points) K is geometric with parameter $p = 1/6$. $\big($*Hint*: The probability mass function (PMF) of geometric random variable with parameter p is $P(K = n) = p(1 - p)^{n-1}$ for $n = 1, 2, 3, \ldots$, and its mean is $E[K] = 1/p$.$\big)$

Solution:

(a) If K is binomial$(6, 1/2)$, the mean of random variable K is $E[K] = np = 3$.

$$P(K < E[K]) = P(K < 3) = \sum_{k=0}^{2} P(K = k) = \binom{6}{0}(\frac{1}{2})^6 + \binom{6}{1}(\frac{1}{2})(\frac{1}{2})^5 + \binom{6}{2}(\frac{1}{2})^2(\frac{1}{2})^4 = \frac{11}{32}.$$

(b) If K is Poisson(3), the mean of random variable K is $E[K] = \lambda = 3$.

$$P(K < E[K]) = P(K < 3) = \sum_{k=0}^{2} P(K = k) = \sum_{k=0}^{2} \frac{3^k e^{-3}}{k!} = (1 + 3 + \frac{9}{2})e^{-3} = \frac{17}{2}e^{-3}.$$

(c) If K is geometric$(1/6)$, the mean of random variable K is $E[K] = 1/p = 6$. Then

$$P(K < E[K]) = 1 - P(K \geq E[K]) = 1 - \big(P(K = E[K]) + P(K > E[K])\big).$$

$$P(K = E[K]) = P(K = 6) = (1 - p)^5 p = (\frac{5}{6})^5 \frac{1}{6},$$

$$\begin{aligned} P(K > E[K]) = P(K > 6) &= \sum_{k=7}^{\infty} (1 - p)^{k-1} p = p\big((1 - p)^6 + (1 - p)^7 + \cdots\big) \\ &= p(1 - p)^6\big(1 + (1 - p) + (1 - p)^2 + \cdots\big) = (1 - p)^6 = (\frac{5}{6})^6. \end{aligned}$$

Therefore,

$$P(K < E[K]) = 1 - (\frac{5}{6})^5 = \frac{4651}{7776} = 0.598.$$

Problem 4. (10 points) How many positive integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 15$, where $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4$?

Solution:
Let $y_1 = x_1 - 1$, $y_2 = x_2 - 2$, $y_3 = x_3 - 3$, $y_4 = x_4 - 4$. Then we can reformulate the problem as follows,

$$y_1 + y_2 + y_3 + y_4 = 5,$$

where $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0$. It's can be viewed as a problem of the number of distinguishable permutations of $n = 8 = n_1 + n_2$ objects of $k = 2$ different types, where $n_1 = 5$ for type-1 and $n_2 = 3$ for type-2 objects partition all the type-1 objects into $n_2 + 1 = 4$ groups, say group i, $i = 1, \ldots, 4$. Let y_i denote the number of type-1 objects in group i. Then we can obtain a solution to the problem. For instance, a permutation shows ABABAAAB (8 objects of 2 types, denoted as "A" and "B" respectively), and then the corresponding solution is $(y_1, y_2, y_3, y_4) = (1, 1, 3, 0)$. Therefore, by Theorem 2.4, the total number of solutions is given by

$$\frac{n!}{n_1!n_2!} = \frac{8!}{3!5!}.$$

Problem 5. (15 points) Determine the following statements are True or False.

(a) (10 points) If $E[XY] = E[X]E[Y]$, then X and Y are independent.

(b) (5 points) If $P(A \cap B \cap C) = P(A)P(B)P(C)$, then A, B and C are mutually independent.

Solution:

(a) False. Let us prove this by a counterexample. Consider $X \in \{-1, 0, 1\}$ and $P(X = -1) = P(X = 0) = P(X = 1) = \frac{1}{3}$. And let $Y = X^2$, we have

$$\begin{cases} P(Y = 1) = \frac{2}{3}, \\ P(Y = 0) = \frac{1}{3}. \end{cases}$$

Then,

$$\begin{aligned} E[XY] - E[X]E[Y] &= E[X^3] - E[X]E[Y] \\ &= ((-1)^3 \cdot \frac{1}{3} + 0^3 \cdot \frac{1}{3} + 1^3 \cdot \frac{1}{3}) - (0 \cdot \frac{2}{3}) \\ &= 0 - 0 = 0. \end{aligned}$$

However,

$$P(X = 1 \cap Y = 0) = 0, \ P(X = 1) = \frac{1}{3} \text{ and } P(Y = 0) = \frac{1}{3}$$

$$\Rightarrow P(X = 1 \cap Y = 0) \neq P(X = 1)P(Y = 0),$$

which implies X and Y are not independent.

(b) False. We can prove this by a counterexample. Given $S = \{1, 2, 3, 4\}$ and each number with equal probability. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \emptyset$. Then, we have

$$P(A \cap B \cap C) = P(\emptyset) = 0 = P(A)P(B)P(C).$$

Nevertheless,

$$P(A \cap B) = P(\{2, 3\}) = \frac{2}{4} \neq \frac{3}{4} \cdot \frac{3}{4} = P(A)P(B),$$

which implies A and B are not independent. Hence, the statement is false.

Problem 6. (10 points) Passengers are making reservations for a particular flight on a small commuter plane 24 hours a day at a Poisson rate of 2 reservations per 6 hours. If 30 seats are available for the flight, what is the probability that by the end of the third day all the plane seats are reserved? (No need to calculate the exact value.)

Solution:
Let X be the number of requests for reservations at the end of the third day. Assume that X is a Poisson distribution with parameter $\lambda = 24$. The probability can be written as

$$P(X \geq 30) = 1 - \sum_{i=0}^{29} P(X = i) = 1 - \sum_{i=0}^{29} \frac{e^{-24}(24)^i}{i!}.$$

Problem 7. (10 points) Five customers enter a wireless corporate store to buy smartphones. If the probability that at least two of them purchase an Android smartphone is 0.5 and the probability that all of them buy non-Android smartphones is 0.23, what is the probability that exactly one Android smartphone is purchased?

Solution:
For $0 \leq i \leq 5$, let A_i be the event that exactly i of these customers buy an Android smartphone. We have that

$$P(A_0) + P(A_1) + \sum_{i=2}^{5} P(A_i) = 1.$$

This implies that $0.23 + P(A_1) + 0.5 = 1$. Solving this for $P(A_1)$, we obtain $P(A_1) = 0.27$.

Problem 8. (10 points) An ordinary deck of 52 cards is well-shuffled, and then the cards are turned face up one by one until an face card (J,Q,K) appears. Find the expected number of cards that are face up. (No need to calculate the exact value.)

Solution:
Let X be the number of cards to be turned face up until an face card appears. Let A be the event that no face card appears among the first $i - 1$ cards that are turned face up. Let B be the event that the i-th card turned face up is a face card. We have

$$P(X = i) = P(A \cap B) = P(B|A)P(A) = \frac{12}{52 - (i - 1)} \frac{\binom{40}{i-1}}{\binom{52}{i-1}}.$$

Therefore,

$$E(X) = \sum_{i=1}^{41} i \cdot P(X = i).$$

Problem 9. (20 points) Suppose each day (starting on day 1) you buy one lottery ticket with probability 1/2; otherwise, you buy no tickets. A ticket is a winner with probability p independent of the outcome of all other tickets. Let N_i be the event that on day i you do not buy a ticket. Let W_i be the event that on day i you buy a winning ticket. Let L_i be the event that on day i you buy a losing ticket.

(a) (5 points) What are $P(W_{33})$, $P(L_{87})$ and $P(N_{99})$?

(b) (5 points) Let K be the number of the day on which you buy your first lottery ticket. Find the probability mass function (PMF) $P(K = k)$.

(c) (5 points) Find the PMF of R, the number of lossing lottery tickets you have purchased in m days.

(d) (5 points) Let D be the number of the day on which you buy your j-th losing ticket. What is $P(D = d)$?

Solution:

(a) Since each day is independent of any other day, $P(W_{33})$ is just the probability that a winning lottery ticket was bought. Similarly for $P(L_{87})$ and $P(N_{99})$ become just the probability that a losing ticket was bought and that no ticket was bought on a single day, respectively. Therefore,

$$P(W_{33}) = \frac{p}{2}, \;\; P(L_{87}) = \frac{(1-p)}{2}, \;\; P(N_{99}) = \frac{1}{2}.$$

(b) Suppose we say a success occurs on the kth trial if on day k we buy a ticket. Otherwise, a failure occurs. The probability of success is simply 1/2. The random variable K is just the number of trials until the first success and has the geometric PMF

$$P(K = k) = \begin{cases} (1/2)(1/2)^{k-1} = (1/2)^k & k = 1, 2, ... \\ 0 & \text{otherwise.} \end{cases}$$

(c) The probability that you decide to buy a ticket and it is a losing ticket is $(1-p)/2$, independent of any other day. If we view buying a losing ticket as a Bernoulli success, R, the number of losing lottery tickets bought in m days, has the binomial PMF

$$P(R = r) = \begin{cases} \binom{m}{r} (\frac{1-p}{2})^r (\frac{1+p}{2})^{m-r} & r = 1, 2, ..., m \\ 0 & \text{otherwise.} \end{cases}$$

where $(1+p)/2$ is the probability of buying a winning ticket or that you didn't buy ticket on a single day.

(d) Letting D be the day on which the jth losing ticket is bought, we can find the probability that $D = d$ by noting that $j-1$ losing tickets must have been purchased in the $d-1$ previous days. Therefore D has the Negative binomial PMF

$$P(D = d) = \begin{cases} \binom{d-1}{j-1} (\frac{1-p}{2})^j (\frac{1+p}{2})^{d-j} & d = j, j+1, ... \\ 0 & \text{otherwise.} \end{cases}$$

Problem 10. (10 points) Derive the PMF of Poisson random variable with parameter λ by approximation of the Binomial PMF with parameters (n, p) when the number of trials is large $(n \to \infty)$. (*Hint*: $(1-\lambda/n)^n \to e^{-\lambda}$ as $n \to \infty$).
Solution:
Let X be a Binomial random variable with parameters (n, p), and let its mean value $np = \lambda$, where λ is also the mean value of Poisson random variable. Then

$$\begin{aligned} P(X = i) &= \binom{n}{i} p^i (1-p)^{n-i} \\ &= \frac{n!}{(n-i)!i!} (\frac{\lambda}{n})^i (1 - \frac{\lambda}{n})^{n-i} \\ &= \frac{n(n-1)(n-2)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^i}. \end{aligned}$$

Now, for large n and appreciable λ, $(1-\lambda/n)^i$ approximates 1, $(1-\lambda/n)^n \to e^{-\lambda}$ as $n \to \infty$, and $[n(n-1)(n-2)\cdots(n-i+1)]/n^i$ approximates 1, because its numerator and denominator are both polynomials in n of degree i. Thus as $n \to \infty$,

$$P(X = i) \to \frac{e^{-\lambda}\lambda^i}{i!}.$$