EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Final Exam

Coverage: Chapters 6–11

Date: May 30, 2023

Time: 13:20 – 15:10

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<u>**Notice**</u>:

1. *Write your name, student ID number on both problem sheets and answer sheet. You need to return both sheets before/at the end of the examination.*
2. *There are 8 questions with total 110 points.*
3. *This is a* **closed book examination**.
4. *You need to provide clear logical reasoning for your answer.*

Problem 1. (15 points) Let the joint probability density function (PDF) of random variables X and Y be

$$f_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 points) Determine the value of c.

(b) (6 points) Find the marginal PDF $f_X(x)$ and calculate $E[X]$.

(c) (5 points) Find the conditional PDF $f_{Y|X}(y|x)$ and $E[Y|X=1]$.

Solution:

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^1\int_0^2 c(x^2+y^2)dxdy = \frac{10}{3}c = 1 \implies c = \frac{3}{10}.$$

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 \frac{3}{10}(x^2+y^2)dy = \begin{cases} \frac{3}{10}x^2 + \frac{1}{10} & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^2 x(\frac{3}{10}x^2 + \frac{1}{10})dx = \frac{7}{5}.$$

(c)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{0.3(x^2+y^2)}{0.3x^2+0.1} = \begin{cases} \frac{3(x^2+y^2)}{3x^2+1} & 0 \le x \le 2,\ 0 \le y \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

$$E(Y|X=1) = \int_0^1 y\frac{3+3y^2}{4}dy = \frac{9}{16}.$$

Problem 2. (15 points) Let the PDF of a random variable X be $f_X(x) = \frac{x^n}{n!}e^{-x}$, $x \ge 0$. Show that $P(0 < X < 2n+2) > \frac{n}{n+1}$. $(Hint : \int_0^\infty x^n e^{-x}dx = n!)$

Solution:
The variance of X can be calculated by $\mathrm{Var}[X] = E[X^2] - E[X]^2$, and

$$E[X] = \int_0^\infty x f_X(x)dx = \int_0^\infty \frac{x^{n+1}}{n!}e^{-x}dx = \frac{(n+1)!}{n!} = n+1,$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)dx = \int_0^\infty \frac{x^{n+2}}{n!}e^{-x}dx = \frac{(n+2)!}{n!} = (n+1)(n+2),$$

$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = (n+2)(n+1) - (n+1)^2.$$

According to Chebyshev's inequality,

$$P(|X - E[X]| \ge t) \le \frac{\mathrm{Var}[X]}{t^2},$$

then the probability can be bounded by

$$P(0 < X < 2n+2) = P\big(|X-(n+1)| < n+1\big) > 1 - \frac{\mathrm{Var}[X]}{(n+1)^2}$$
$$= 1 - \frac{(n+2)(n+1)-(n+1)^2}{(n+1)^2} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Problem 3. (10 points)

(a) (5 points) Prove that $E[X] = \int_0^\infty [1-F(x)]dx - \int_{-\infty}^0 F(x)dx$, where $F(x)$ is the cumulative distribution function (CDF) of X.

(b) (5 points) Let the PDF of a random variable X be

$$f_X(x) = \begin{cases} \frac{1}{3} & -1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the mean of X by the result of (a).

Solution:

(a) According to the integration by parts, we have

$$\int_0^\infty [1-F(x)]dx = \big[x(1-F(x))\big]_0^\infty + \int_0^\infty x\frac{dF(x)}{dx}dx = \int_0^\infty xf(x)dx. \quad (1)$$
$$\int_{-\infty}^0 xf(x)dx = \int_{-\infty}^0 x\frac{dF(x)}{dx}dx = \big[xF(x)\big]_{-\infty}^0 - \int_{-\infty}^0 F(x)dx = -\int_{-\infty}^0 F(x)dx. \quad (2)$$

Combine the equations (1) and (2), then we have

$$\int_0^\infty [1-F(x)]dx - \int_{-\infty}^0 F(x)dx = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx = E[X],$$

which is the desired result of the problem.

(b) Since X is the uniform random variable, the CDF of X is given by

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{(x-(-1))}{3} & -1 \le x < 2, \\ 1 & 2 \le x. \end{cases}$$

Using the result of (a), the mean of X can be calculated by

$$E[X] = \int_0^2 (1-F(x))dx - \int_{-1}^0 F(x)dx = \int_0^2 (1-\frac{x+1}{3})dx - \int_{-1}^0 \frac{x+1}{3}dx = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}.$$

Problem 4. (15 points) Let $X_1, X_2, \ldots, X_k$ be a sequence of independent identically distributed (i.i.d.) random variables with mean m_X and variance σ_X^2, and let N be an integer-valued random variable independent of the X_k's. The mean and variance of N are m_N and σ_N^2, respectively. Let $S = \sum_{k=1}^N X_k$. Determine the mean and variance of S.

Solution:
First, the mean of S can be calculated by $E[S] = E[E[S|N]]$. Then,

$$E[S|N = n] = E[X_1 + \cdots + X_n|N = n] = nE[X_i] = nm_X,$$

which implies

$$E[S] = E[E[S|N]] = E[Nm_X] = m_X E[N] = m_X m_N.$$

The variance of S can be calculated $\text{Var}[S] = E[S^2] - (E[S])^2$, and $E[S^2] = E[E[S^2|N]]$. Then,

$$E[S^2|N = n] = \text{Var}[S|N = n] + (E[S|N = n])^2 = n\sigma_X^2 + n^2 m_X^2,$$

which implies

$$\begin{aligned}
\text{Var}[S] &= E[S^2] - E[S]^2 \\
&= E[E[S^2|N]] - E[S]^2 \\
&= E[N\sigma_X^2 + N^2 m_X^2] - m_X^2 E[N]^2 \\
&= E[N]\sigma_X^2 + (E[N^2] - E[N]^2) m_X^2 \\
&= m_N \sigma_X^2 + \sigma_N^2 m_X^2.
\end{aligned}$$

Problem 5. (15 points) Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0,\ y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Calculate $P[X > Y]$ and $P[X + Y < 1]$.

(b) (5 points) Find $P[\min(X, Y) \geq 1]$.

(c) (5 points) Find $P[\max(X, Y) \leq 1]$.

Solution:

(a)

$$P[X > Y] = \int_0^\infty \int_0^x 6e^{-(2x+3y)} dy dx = \int_0^\infty (2e^{-2x} - 2e^{-5x}) dx = \frac{3}{5}.$$

$$P[X + Y < 1] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)} dy dx = \int_0^1 2e^{-2x}(1 - e^{-3(1-x)}) dx = 1 + 2e^{-3} - 3e^{-2}.$$

(b)

$$P[\min(X, Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)} dy dx = \int_1^\infty 2e^{-(2x+3)} dx = e^{-5}.$$

(c)

$$P[\max(X, Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)} dy dx = \int_0^1 2e^{-2x}(1 - e^{-3}) dx = (1 - e^{-2})(1 - e^{-3}).$$

Problem 6. (10 points) Let X be a standard normal random variable with PDF $f_X(x) = (1/\sqrt{2\pi})e^{-x^2/2}$, $-\infty < x < \infty$. Calculate $E\left[\dfrac{X}{1+X^2}\right]$.

Solution:
It's easy to verify that $\dfrac{X}{1+X^2}$ is an odd function. Let $g(x) = \dfrac{x}{1+x^2}$, $\forall x \in \mathbb{R}$. Then

$$g(x) = \frac{x}{1+x^2} = -\frac{(-x)}{1+(-x)^2} = -g(-x),\ \forall x \in \mathbb{R}.$$

Moreover, $E[|g(X)|] \leq E[|X|] = 2 < \infty$ implies that $E[g(X)]$ exists. The expectation of $\dfrac{X}{1+X^2}$ can be calculated by

$$E\left[\frac{X}{1+X^2}\right] = \int_{-\infty}^{\infty} \underbrace{g(x)}_{\text{odd}}\ \underbrace{f_X(x)}_{\text{even}}\, dx = 0.$$

Note that an odd function multiplied by an even function is still an odd function, and the last equation is equal to zero due to the antisymmetry of the odd function.

Problem 7. (15 points) There are 3 light bulbs. The lifetimes of the light bulbs are independent and exponentially distributed with different means. Light bulb A, B, and C have mean of lifetimes of 1000, 1500, and 3000 hours respectively. If we turn on the 3 light bulbs simultaneously, what is $P[\text{Light bulb A is the first one to stop working}]$?

Solution:
Let X_1, X_2, X_3 be the exponential random variables with parameters $\lambda_1 = 1/1000$, $\lambda_2 = 1/1500$, and $\lambda_3 = 1/3000$ which represent the lifetime of the light bulbs A, B, and C, respectively. Since X_1, X_2, X_3 are independent, the joint PDF of X_1, X_2, X_3 is

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} \lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2}\lambda_3 e^{-\lambda_3 x_3} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} P(X_1 \leq X_2 \leq X_3) + P(X_1 \leq X_3 \leq X_2) &= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2}\lambda_3 e^{-\lambda_3 x_3}\, dx_3 dx_2 dx_1 \\ &\quad + \int_0^\infty \int_{x_1}^\infty \int_{x_3}^\infty \lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2}\lambda_3 e^{-\lambda_3 x_3}\, dx_2 dx_3 dx_1 \\ &= \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)} + \frac{\lambda_1\lambda_3}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)} \\ &= \frac{\lambda_1}{(\lambda_1+\lambda_2+\lambda_3)} = \frac{1}{2}. \end{aligned}$$

Problem 8. (15 points) Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Note that the probability mass function (PMF) of Poisson random variable is

$$P_X(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Let $W = X_1 + X_2 + \cdots + X_n$. Find the PMF and moment-generating function (MGF) of W.

(b) (10 points) Provided that $\lambda_i = \lambda$ for all i, let $Z_n = (W - n\lambda)/(\sqrt{n\lambda})$. Find the MGF $M_{Z_n}(t)$ using the result in part (a), and then derive the $\lim_{n\to\infty} M_{Z_n}(t)$ and the associated PDF. What does your result justify? ($Hint : e^x = \sum_{n=0}^{\infty} x^n/n!$)

Solution:

(a) Since $X_1 \sim \text{Poisson}(\lambda_1)$, and by the definition of the MGF, we have

$$M_{X_1}(t) = E[e^{tX_1}] = \sum_{x_1=0}^{\infty} e^{tx_1} \frac{e^{-\lambda_1}\lambda_1^{x_1}}{x_1!} = e^{-\lambda_1} \sum_{x_1=0}^{\infty} \frac{(\lambda_1 e^t)^{x_1}}{x_1!} = e^{-\lambda_1} e^{\lambda_1 e^t} = e^{\lambda_1(e^t-1)}.$$

The MGF of W is given by

$$\begin{aligned} M_W(t) &= E[e^{tW}] = E[e^{tX_1+tX_2+\cdots+tX_n}] \\ &= E[e^{tX_1}]E[e^{tX_2}]\cdots E[e^{tX_n}] \qquad \text{(Since } X_1, X_2, \ldots, X_n \text{ are independent)} \\ &= M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t) \\ &= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}. \end{aligned}$$

Hence, W is also a Poisson random variable with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. In other words, its PMF is given by

$$P_W(w) = \frac{e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)}(\lambda_1+\lambda_2+\cdots+\lambda_n)^w}{w!},\ w = 0, 1, 2 \ldots.$$

(b) By the result of (a), we have

$$M_{Z_n}(t) = E[e^{tZ_n}] = E[e^{t(\frac{W-n\lambda}{\sqrt{n\lambda}})}] = e^{-\sqrt{n\lambda}t} E[e^{\frac{tW}{\sqrt{n\lambda}}}] = e^{-\sqrt{n\lambda}t} e^{n\lambda(e^{\frac{t}{\sqrt{n\lambda}}}-1)} = e^{n\lambda(e^{\frac{t}{\sqrt{n\lambda}}}-1-\frac{t}{\sqrt{n\lambda}})}.$$

Moreover, as $n \to \infty$, we have

$$\begin{aligned} M_Z(t) \triangleq \lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty} e^{n\lambda(e^{\frac{t}{\sqrt{n\lambda}}}-1-\frac{t}{\sqrt{n\lambda}})} \\ &= \lim_{n\to\infty} e^{n\lambda\left[(1+\frac{t}{\sqrt{n\lambda}}+\frac{t^2}{2!(\sqrt{n\lambda})^2}+\frac{t^3}{3!(\sqrt{n\lambda})^3}+\cdots)-1-\frac{t}{\sqrt{n\lambda}}\right]} \\ &= e^{t^2/2}, \end{aligned}$$

which is exactly the MGF of a standard normal random variable. Therefore, the PDF of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},\ -\infty < z < \infty.$$

The above results verify the central limit theorem, that is, Z_n, the normalized sample mean of i.i.d. non-normal $\{X_1, X_2, \ldots, X_n\}$ converges to the standard normal random variable Z in distribution.