EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Homework #5 – Solutions
Coverage: May 23, 2023
Due date: Chapters 8–9

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Problem 8.1.3. (12.5 points) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} k(x^2+y^2) & \text{if } (x,y) = (1,1),\ (1,3),\ (2,3), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of the constant k.

(b) Determine the marginal probability mass functions of X and Y.

(c) Find $E(X)$ and $E(Y)$.

Solution:

(a) $k\big[(1+1)+(1+9)+(4+9)\big] = 1$ implies that $k = 1/25$.

(b)

$$p_X(1) = p(1,1) + p(1,3) = 12/25,\ p_X(2) = p(2,3) = 13/25.$$
$$p_Y(1) = p(1,1) = 2/25,\ p_Y(3) = p(1,3) + p(2,3) = 23/25.$$

Therefore,

$$p_X(x) = \begin{cases} 12/25 & \text{if } x = 1, \\ 13/25 & \text{if } x = 2. \end{cases}$$

$$p_Y(y) = \begin{cases} 2/25 & \text{if } y = 1, \\ 23/25 & \text{if } y = 3. \end{cases}$$

(c)

$$E(X) = 1 \cdot \frac{12}{25} + 2 \cdot \frac{13}{25} = \frac{38}{25},\ E(Y) = 1 \cdot \frac{2}{25} + 3 \cdot \frac{23}{25} = \frac{71}{25}.$$

Problem 8.1.11. (12.5 points) Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the marginal probability density functions of X and Y, respectively.

(b) Calculate $E(X)$ and $E(Y)$.

Solution:

(a)

$$f_X(x) = \int_0^x 8xy\,dy = 4x^3, \quad 0 \le x \le 1,$$
$$f_Y(y) = \int_y^1 8xy\,dx = 4y(1-y^2), \quad 0 \le y \le 1.$$

(b)

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x \cdot 4x^3 dx = \frac{4}{5},$$
$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y \cdot 4y(1-y^2)dy = \frac{8}{15}.$$

Problem 8.2.17. (12.5 points) Let X and Y be independent random points from the interval $(-1,1)$. Find $E[\max(X,Y)]$.

Solution:
Let F and f be the distribution and probability density functions of $\max(X,Y)$, respectively. Clearly,

$$F(t) = P[\max(X,Y) \le t] = P(X \le t,\, Y \le t) = P(X \le t)P(Y \le t) = (\frac{t+1}{2})^2.$$

Thus

$$f(t) = F'(t) = \frac{t+1}{2}, \quad -1 < t < 1.$$

Therefore,

$$E[\max(X,Y)] = \int_{-1}^1 t \cdot (\frac{t+1}{2})\,dt = \frac{1}{3}.$$

Problem 8.3.10. (12.5 points) First a point Y is selected at random from the interval $(0,1)$. Then another point X is selected at random from the interval $(Y,1)$. Find the probability density function of X.

Solution:
Let $f(x,y)$ be the joint probability density function of X and Y. Clearly,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\,dy.$$

Now

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1-y} & 0 < y < 1, \quad y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$f_X(x) = \int_0^x \frac{1}{1-y}\,dy = -\ln(1-x), \quad 0 < x < 1.$$

and hence

$$f_X(x) = \begin{cases} -\ln(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 9.1.14 (12.5 points) Let $X_1, X_2, ..., X_n$ be identically distributed, independent, exponential random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that

$$E[\min(X_1, ..., X_n)] < \min\{E(X_1), ..., E(X_n)\}.$$

Solution:
We know that

$$P(X_i \geq t) = e^{-\lambda_i t}, \text{ for } i = 1, ..., n,\ t \geq 0.$$

Therefore,

$$\begin{aligned} P[\min(X_1, X_2, ..., X_n) > t] &= P(X_1 > t, X_2 > t, ..., X_n > t) \\ &= P(X_1 > t)P(X_2 > t) \cdots P(X_n > t) \\ &= (e^{-\lambda_1 t})(e^{-\lambda_2 t}) \cdots (e^{-\lambda_n t}) \\ &= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}, \end{aligned}$$

which implies $\min(X_1, X_2, ..., X_n)$ is a exponential random variable with parameter $(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$. The inequality follows from the fact that for $i = 1, 2, ..., n$,

$$E[\min(X_1, ..., X_n)] = \frac{1}{\lambda_1 + \cdots + \lambda_n} < \frac{1}{\lambda_i} = E(X_i).$$

Problem 9.2.4. (12.5 points) Let X_1, X_2, X_3, and X_4 be independent exponential random variables, each with parameter λ. Find $P(X_{(4)} \geq 3\lambda)$. (*Hint* : By the theorem 9.5, $X_{(n)}$ is the order statistics of the independent and identically distributed continuous random variables $X_1, X_2, \ldots, X_n$.)

Solution:
By theorem 9.5, the probability density function f_4 of $X_{(4)}$ is given by

$$f_4(x) = \frac{4!}{3!\,0!}\lambda e^{-\lambda x}(1 - e^{-\lambda x})^3 (e^{-\lambda x})^{4-4} = 4\lambda e^{-\lambda x}(1 - e^{-\lambda x})^3.$$

The desired probability is

$$\int_{3\lambda}^{\infty} 4\lambda e^{-\lambda x}(1 - e^{-\lambda x})^3\,dx = 1 - (1 - e^{-3\lambda^2})^4.$$

Problem 9.3.5. (12.5 points) Of the drivers who are insured by a certain insurance company and get into at least one accident during a random year, 15% are low-risk drivers, 35% are moderate-risk drivers, and 50% are high-risk drivers. Suppose that drivers insured by this company get into an accident independently of each other. If an actuary randomly chooses five drivers insured by this company, who got into an accident one or more times last year, what is the probability that there are at least two more high-risk drivers among them than the low-risk and moderate-risk drivers combined?

Solution:

$$\begin{aligned} &P(X_1 = 0, X_2 = 0, X_3 = 5) + P(X_1 = 0, X_2 = 1, X_3 = 4) + P(X_1 = 1, X_2 = 0, X_3 = 4) \\ &= \frac{5!}{0!\,0!\,5!}(0.15)^0(0.35)^0(0.50)^5 + \frac{5!}{0!\,1!\,4!}(0.15)^0(0.35)^1(0.50)^4 + \frac{5!}{1!\,0!\,4!}(0.15)^1(0.35)^0(0.50)^4 \\ &\approx 0.19. \end{aligned}$$

Problem Ch9. Review 4. (12.5 points) The joint probability density function of random variables X, Y, and Z is given by

$$f(x,y,z) = \begin{cases} c(x+y+2z) & 0 < x,\, y,\, z < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Find $P(X < 1/3 \,|\, Y < 1/2,\, Z < 1/4)$.

Solution:

(a)

$$c \int_0^1 \int_0^1 \int_0^1 (x+y+2x)\, dzdydx = 1 \implies c = 1/2.$$

(b)

$$\begin{aligned} P(X < 1/3 \,|\, Y < 1/2,\, Z < 1/4) &= \frac{P(X < 1/3,\, Y < 1/2,\, Z < 1/4)}{P(Y < 1/2,\, Z < 1/4)} \\ &= \frac{\displaystyle\int_0^{1/3} \int_0^{1/2} \int_0^{1/4} \frac{1}{2}(x+y+2z)\, dzdydx}{\displaystyle\int_0^{1} \int_0^{1/2} \int_0^{1/4} \frac{1}{2}(x+y+2z)\, dzdydx} \\ &= \frac{1/36}{1/8} = \frac{2}{9}. \end{aligned}$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)