EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Homework #4 – Solutions
Coverage: Chapters 6 and 7
Due date: May 9, 2023

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Problem 6.1.12. (12.5 points) Let X denote the lifetime of a radio, in years, manufactured by a certain company. The probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{15}e^{-x/15} & 0 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that, of eight such radios, at least four last more than 15 years?

Solution:

$$P(X > 15) = \int_{15}^{\infty} \frac{1}{15}e^{-x/15}dx = \frac{1}{e}.$$

Thus the answer is

$$\sum_{i=4}^{8} \binom{8}{i} \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right)^{8-i} = 0.3327.$$

Problem 6.2.8. (12.5 points) Let X be a Cauchy random variable with the probability density function

$$f(x) = \frac{1}{\pi(1+x^2)} \quad -\infty < x < \infty.$$

Find the probability density function of $Z = \arctan X$.

Solution:
Let G and g be the distribution and probability density functions of Z, respectively. Since Z is a strictly increasing function of X, for $-\pi/2 < z < \pi/2$,

$$\begin{aligned} G(z) = P(\arctan X \leq z) = P(X \leq \tan z) &= \int_{-\infty}^{\tan z} \frac{1}{\pi(1+x^2)}\,dx \\ &= \left[\frac{1}{\pi}\arctan x\right]_{-\infty}^{\tan z} = \frac{1}{\pi}z + \frac{1}{2}. \end{aligned}$$

Thus

$$g(z) = \frac{dG(z)}{dz} = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < z < \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6.3.14. (12.5 points) Suppose that X, the interarrival time between two customers entering a certain post office, satisfies

$$P(X > t) = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \quad t \geq 0,$$

where $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$, $\lambda > 0$, $\mu > 0$. Calculate the expected value of X.

Solution:
By Remark6.4,

$$E(X) = \int_0^\infty P(X > t)\,dt = \int_0^\infty (\alpha e^{-\lambda t} + \beta e^{-\mu t})\,dt = \frac{\alpha}{\lambda} + \frac{\beta}{\mu}.$$

Problem 6.3.16. (12.5 points) Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{1}{\pi} x \sin x & 0 \le x < \pi \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$E(X^{n+1}) + (n+1)(n+2)E(X^{n-1}) = \pi^{n+1}.$$

Solution:
Using integration by parts twice, we obtain

$$\begin{aligned} E(X^{n+1}) &= \frac{1}{\pi}\int_0^\pi x^{n+2}\sin x dx \\ &= \pi^{n+1} + (n+2)\frac{1}{\pi}\int_0^\pi x^{n+1}\cos x dx \\ &= \pi^{n+1} + (n+2)\left[-(n+1)\frac{1}{\pi}\int_0^\pi x^n \sin x dx\right] \\ &= \pi^{n+1} + (n+2)\left[-(n+1)E(X^{n-1})\right]. \end{aligned}$$

Hence

$$E(X^{n+1}) + (n+1)(n+2)E(X^{n-1}) = \pi^{n+1}.$$

Problem 7.1.6. (12.5 points) A point is selected at random on a line segment of length ℓ. What is the probability that the longer segment is at least twice as long as the shorter segment?

Solution:
The problem is equivalent to choosing a random X from $(0, \ell)$. The desired probability is

$$P(X \le \frac{\ell}{3}) + P(X \ge \frac{2\ell}{3}) = \frac{\ell/3}{\ell} + \frac{\ell - (2\ell/3)}{\ell} = \frac{2}{3}.$$

Problem 7.2.22. (12.5 points) Let $\alpha \in (-\infty, \infty)$ and $Z \sim N(0,1)$. Find $E(e^{\alpha Z})$.

Solution:
We have

$$\begin{aligned} E(e^{\alpha Z}) &= \int_{-\infty}^\infty e^{\alpha z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\ &= e^{\alpha^2/2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\alpha^2 + \alpha z - \frac{1}{2}z^2}dz \\ &= e^{\alpha^2/2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\alpha)^2}dz \\ &= e^{\alpha^2/2}. \end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\alpha)^2}$ is the probability density function of a normal variable with mean α and variance 1, $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\alpha)^2}dz = 1$.

Problem 7.3.6. (12.5 points) Let X be an exponential random variable with parameter λ. Find

$$P(|X - E(X)| \ge 2\sigma_X).$$

Solution:

$$
\begin{aligned}
P\big(|X-E(X)|\ge 2\sigma_X\big) &= P\big(|X-\frac{1}{\lambda}|\ge \frac{2}{\lambda}\big)\\
&= P\big(X-\frac{1}{\lambda}\ge \frac{2}{\lambda}\big)+P\big(X-\frac{1}{\lambda}\le -\frac{2}{\lambda}\big)\\
&= P\big(X\ge \frac{3}{\lambda}\big)+P\big(X\le -\frac{1}{\lambda}\big)\\
&= e^{-\lambda(3/\lambda)}+0=e^{-3}=0.049787.
\end{aligned}
$$

Problem 7.4.4. (12.5 points) Let X be the time to failure of a mobile robot that is used as an automated guide vehicle. Suppose that $E(X)=7$ years, and X has a gamma distribution with $r=4$. Find $Var(X)$ and $P(5\le X\le 7)$.

Solution:
Suppose that the parameters of the gamma distribution are $r=4$ and λ. So $E(X)=r/\lambda$ implies that $\lambda=4/7$. Therefore, $Var(X)=r/\lambda^2=4/(16/49)=49/4=12.25$. To find $P(5\le X\le 7)$, note that the probability density function of X is

$$
f(x)=\begin{cases}\dfrac{(4/7)^4x^3e^{-4x/7}}{\Gamma(4)}=\dfrac{1}{3!}\cdot\big(\dfrac{4}{7}\big)^4x^3e^{-4x/7} & x\ge 0,\\ 0 & \text{otherwise.}\end{cases}
$$

Applying integraton by parts, we have

$$
\begin{aligned}
P(5\le X\le 7) &= \frac{1}{6}\big(\frac{4}{7}\big)^4\int_5^7 x^3e^{-4x/7}\,dx\\
&= \frac{1}{6}\big(\frac{4}{7}\big)^4\left[\frac{1}{256}e^{-4x/7}(-448x^3-2352x^2-8232x-14406)\right]_5^7\\
&\approx 0.246.
\end{aligned}
$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)