EE306002: Probability Department of Electrical Engineering National Tsing Hua University

Homework #3 – Solutions Coverage: Chapter 5

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Problem 5.1.24 Vincent is a patient with the life threatening blood cancer leukemia, and he is in need of a bone marrow transplant. He asks n people whether or not they are willing to donate bone marrow to him if they are a close bone marrow match for him. Suppose that each person's response, independently of others, is positive with probability p_1 and negative with probability $1 - p_1$. Furthermore, suppose that, independently of the others, the probability is p_2 that a person tested is a close match. Find the probability that Vincent finds at least one close bone marrow match among these n people.

(*Hint*: For $0 \le i \le n$, let A_i be the event that *i* of the *n* individuals respond positively to Vincent's request. Let X be the number of close bone marrow matches among those who will be tested. To find $P(X \ge 1) = 1 - P(X = 0)$, calculate P(X = 0) by using the Law of Total Probability.)

Solution:

By the law of Total Probability, we have

$$P(X = 0) = \sum_{i=0}^{n} P(X = 0 | A_i) P(A_i)$$

=
$$\sum_{i=0}^{n} (1 - p_2)^i {\binom{n}{i}} p_1^i (1 - p_1)^{n-i}$$

=
$$\sum_{i=0}^{n} {\binom{n}{i}} [p_1(1 - p_2)]^i (1 - p_1)^{n-i}$$

=
$$\sum_{i=0}^{n} {\binom{n}{i}} [p_1 - p_1 p_2]^i (1 - p_1)^{n-i}$$

=
$$(p_1 - p_1 p_2 + 1 - p_1)^n = (1 - p_1 p_2)^n.$$

The next to last equality follows by Binomial Expansion. Then,

$$P(X \ge 1) = 1 - P(X = 0) = 1 - (1 - p_1 p_2)^n$$

Problem 5.3.14 For certain software, independently of other users, the probability is 0.07 that a user encounters a fault. What are the chances that the 30th user is the 5th person encountering a fault?

Solution:

Let X be the number of users until the 5th person encounters a fault. Thus X is negative binomial distribution given by

$$P(X=n) = {\binom{n-1}{i-1}} p^5 (1-p)^{30-5} = {\binom{30-1}{5-1}} (0.07)^5 (1-0.07)^{30-5} = 0.0065.$$

Problem Ch5-Review 2. The time between the arrival of two consecutive customers at a post office is 3 minutes, on average. Assuming that customers arrive in accordance with a Poisson process, find the probability that tomorrow during the lunch hour (between noon and 12:30 p.m.) fewer than seven customers arrive.

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Solution:

Let N(t) be the number of customers arriving at the post office at or prior to t is a Poisson process with $\lambda = 1/3$. Then,

$$P(N(t) \le n) = \sum_{i=0}^{n} P(N(t) = i) = \sum_{i=0}^{6} P(N(30) = i) = \sum_{i=0}^{6} \frac{e^{-(1/3)30}[(1/3)30]^{i}}{i!}.$$

Problem Ch5-Review 20. Passengers are making reservations for a particular flight on a small commuter plane 24 hours a day at a Poisson rate of 3 reservations per 8 hours. If 24 seats are available for the flight, what is the probability that by the end of the second day all the plane seats are reserved?

Solution:

Let X be the number of requests for reservations at the end of the second day. Assume that X is a Poisson distribution with parameter $\lambda = 18$. The probability can be written as

$$P(X \ge n) = 1 - \sum_{i=0}^{n-1} P(X=i) = 1 - \sum_{i=0}^{23} P(X=i) = 1 - \sum_{i=0}^{23} \frac{e^{-18}(18)^i}{i!}.$$

Problem Ch5-Review 26. Suppose that n babies were born at a county hospital last week. Also suppose that the probability of a baby having blonde hair is p. If k of these n babies are blondes, what is the probability that the *i*-th baby born is blonde? (Note: Please show the detail of your proof.)

Solution:

Let A be the event that k of the n babies are bloneds, and let B be the event that the *i*-th baby born is blonde. We have

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{\binom{n-1}{k-1}p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

References

 Saeed Ghahramani, Fundamentals of Probability: With Stochastic Processes, Chapman and Hall/CRC; 4th edition (September 4, 2018)