EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Homework #2 – Solutions

Coverage: Chapters 3 and 4

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Instructor: Chong-Yung Chi

TAs: Chien-Wei Huang & Ting-Jie Huang & Chun-Yi Chen & Sheng-Wei Chou

Problem 3.1.16. (10 points) Prove that if $P(E|F) \geq P(G|F)$ and $P(E|F^c) \geq P(G|F^c)$, then $P(E) \geq P(G)$.

Solution:
The given inequalities imply that $P(EF) \geq P(GF)$ and $P(EF^c) \geq P(GF^c)$. Thus

$$P(E) = P(EF) + P(EF^c) \geq P(GF) + P(GF^c) = P(G).$$

Problem 3.2.5. (10 points) An actuary works for an auto insurance company whose customers all have insured at least one car. She discovers that of all the customers who insure more than one car, 70% of them insure at least one Sport Utility Vehicle (SUV). Furthermore, she observes that 47% of all the customers insure at least one SUV. If 65% of all customers insure more than one car, what percentage of the customers insure exactly one non-SUV?

Solution:
Let E be the event that a randomly selected customer from the auto insurance company has insured more than one car. Let F be the event that he or she has insured at least one SUV. The desired probability is

$$\begin{aligned} P(E^cF^c) &= P((E \cup F)^c) = 1 - P(E \cup F) \\ &= 1 - P(E) - P(F) + P(EF) \\ &= 1 - P(E) - P(F) + P(F|E)P(E) \\ &= 1 - 0.65 - 0.47 + (0.70)(0.65) = 0.335. \end{aligned}$$

Therefore, 33.5% of the customers of this company insure exactly one non-SUV.

Problem 3.3.11. (10 points) When traveling from Springfield, Massachusetts, to Baltimore, Maryland, 30% of the time Charles takes I-91 south and then I-95 south, and 70% of the time he takes I-91 south, then I-84 west followed by I-81 south and I-83 south. Choosing the first option takes Charles a random time between 5 hours and 30 minutes and 10 hours depending upon the traffic. However, choosing the second option takes a random time between 6 hours and 30 minutes and 8 hours. The second option is a longer drive but has much less traffic. What is the probability that on his next trip from Springfield to Baltimore, Charles travels no more than 7 hours and 15 minutes?

Solution:
If Charles chooses option one, his travel time is a random number from the interval (5.5, 10). If he chooses option two, it is a random number from the interval (6.5, 8). Let A be the event that, on his next trip, Charles travels no more than 7 hours and 15 minutes. Let B be the event that he chooses the first option. We have

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= \frac{7.25 - 5.5}{10 - 5.5} \cdot 0.3 + \frac{7.25 - 6.5}{8 - 6.5} \cdot 0.7 = 0.467. \end{aligned}$$

Problem 3.4.14. (10 points) A certain cancer is found in one person in 5000. If a person does have the disease, in 92% of the cases the diagnostic procedure will show that he or she actually has it. If a person does not have the disease, the diagnostic procedure in one out of 500 cases gives a false positive result. Determine the probability that a person with a positive test result has cancer.

Solution:
Let X be the event that a person has cancer and Y be the event that a person has positive test result. Then the probability that a person with a positive result has cancer is $P(X|Y)$. By Baye's formula,

$$\begin{aligned}
P(X|Y) &= \frac{P(XY)}{P(Y)} \\
&= \frac{P(\textit{has cancer and positive result})}{P(\textit{positive result})} \\
&= \frac{P(\textit{has cancer and positive result})}{P(\textit{has cancer and positive result}) + P(\textit{no cancer but positive result})} \\
&= \frac{P(Y|X)P(X)}{P(Y|X)P(X) + P(Y|X^c)P(X^c)} \\
&= \frac{(0.92)(1/5000)}{(0.92)(1/5000) + (1/500)(4999/5000)} \\
&= 0.084.
\end{aligned}$$

Problem 3.5.40. (10 points) A fair coin is tossed n times. Show that the events "at least two heads" and "one or two tails" are independent if $n = 3$ but dependent if $n = 4$.

Solution:
Let A be the event that tossed at least two heads and B be the event that tossed with one or two tails.
For $n = 3$, the probabilities of the given events, respectively, are

$$P(A \mid n = 3) = \binom{3}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} + \binom{3}{3} \cdot (\frac{1}{2})^3 = \frac{1}{2},$$

and

$$P(B \mid n = 3) = \binom{3}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^2 + \binom{3}{2} \cdot (\frac{1}{2})^2 \times \frac{1}{2} = \frac{3}{4}.$$

The probability of their joint occurrence is

$$P(AB \mid n = 3) = \binom{3}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = P(A \mid n = 3) \cdot P(B \mid n = 3).$$

So the given events are independent.
For $n = 4$, the probabilities of the given events, respectively, are

$$P(A \mid n = 4) = 1 - P(A^c \mid n = 4) = 1 - \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 - \binom{4}{0} \cdot (\frac{1}{2})^4 = \frac{11}{16},$$

and

$$P(B \mid n = 4) = \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 + \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^2 = \frac{10}{16}.$$

The probability of their joint occurrence is

$$P(AB \mid n = 4) = \binom{4}{1} \cdot \frac{1}{2} \cdot (\frac{1}{2})^3 + \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^2 = \frac{10}{16} \neq \frac{11}{16} \cdot \frac{10}{16} = P(A \mid n = 4) \cdot P(B \mid n = 4).$$

So the given events are *not* independent.

Problem 4.2.16. (10 points) In a small town there are 40 taxis, numbered 1 to 40. Three taxis arrive at random at a station to pick up passengers. What is the probability that the number of at least one of the taxis is less than 5 ?

Solution:
Let X be the minimum of the three numbers. Then

$$P(X < 5) = 1 - P(X \geq 5) = 1 - \frac{\binom{36}{3}}{\binom{40}{3}} = 0.277.$$

Problem 4.3.5. (10 points) Let X be the number of random numbers selected from $\{0, 1, 2, \ldots, 9\}$ independently until 0 is chosen. Find the probability mass functions of X and $Y = 2X + 1$.

Solution:
Let p be the PMF of X and q be the PMF of Y. We have

$$p(i) = \frac{1}{10} \cdot (\frac{9}{10})^{i-1}, \quad i = 1, 2, \ldots,$$

$$q(j) = P(Y = j) = P(X = \frac{j-1}{2}) = \frac{1}{10} \cdot (\frac{9}{10})^{(j-3)/2}, \quad j = 3, 5, 7, \ldots.$$

Problem 4.4.19. (10 points) An ordinary deck of 52 cards is well-shuffled, and then the cards are turned face up one by one until an ace appears. Find the expected number of cards that are face up.

Solution:
Let X be the number of cards to be turned face up until an ace appears. Let A be the event that no ace appears among the first $i - 1$ cards that are turned face up. Let B be the event that the i-th card turned face up is an ace. We have

$$P(X = i) = P(AB) = P(B|A)P(A) = \frac{4}{52 - (i-1)} \frac{\binom{48}{i-1}}{\binom{52}{i-1}}.$$

Therefore,

$$E(X) = \sum_{i=1}^{49} i \cdot P(X = i) = 10.6.$$

Problem 4.5.6. (10 points) Let X be a discrete random variable with the set of possible values $\{x_1, x_2, \ldots, x_n\}$; X called a **discrete uniform random variable** if

$$P(X = x_i) = \frac{1}{n},\ 1 \leq i \leq n.$$

Find $E(X)$ and $Var(X)$ for the special case, where $x_i = i,\ 1 \leq i \leq n$. Note that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution:

$$\begin{aligned}
E(X) &= \sum_{i=1}^{n}(i \cdot \frac{1}{n}) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}, \\
Var(X) &= E(X^2) - [E(X)]^2 = \sum_{i=1}^{n}(i^2 \cdot \frac{1}{n}) - (\frac{n+1}{2})^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 \\
&= \frac{n+1}{2}(\frac{2n+1}{3} - \frac{n+1}{2}) = \frac{n+1}{2} \cdot \frac{n-1}{6} = \frac{n^2-1}{12}.
\end{aligned}$$

Problem 4.6.1 (10 points) Mr. Norton owns two appliance stores. In store 1 the number of TV sets sold by a salesperson is, on average, 13 per week with a standard deviation of 5. In store 2 the number of TV sets sold by a salesperson is, on average, 7 with a standard deviation of 4. Mr. Norton has a position open for a person to sell TV sets. There are two applicants. Mr. Norton asked one of them to work in store 1 and the other in store 2, each for one week. The salesperson in store 1 sold 10 sets, and the salesperson in store 2 sold 6 sets. Based on this information, which person should Mr. Norton hire?

Solution:
Let X_1 be the number of TV sets the salesperson in store 1 sells and X_2 be the number of TV sets the salesperson in store 2 sells. After standardizing X_1 and X_2, we have $X_1^* = (10-13)/5 = -0.6$ and $X_2^* = (6-7)/4 = -0.25$. Therefore, the number of TV sets the salesperson in store 1 sells is 0.6 standard deviations below the mean, whereas the number of TV sets the salesperson in store 2 sells is 0.25 standard deviations below the mean. So Mr. Norton should hire the salesperson who worked in store 2.

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)