EE306002: Probability **Spring 2023**
Department of Electrical Engineering
National Tsing Hua University

Homework #1 – Solutions
Coverage: Chapters 1 and 2
Due date: March 16, 2023

Instructor: Chong-Yung Chi

TAs: Chien-Wei Huang & Ting-Jie Huang & Chun-Yi Chen & Sheng-Wei Chou

Problem 1.2.5. (10 points) A deck of six cards consists of three black cards numbered 1, 2, 3, and three red cards numbered 1, 2, 3. First, Vann draws a card at random and without replacement. Then Paul draws a card at random and without replacement from the remaining cards. Let A be the event that Paul's card has a larger number than Vann's card. Let B be the event that Vann's card has a larger number than Paul's card.

(a) Are A and B mutually exclusive?

(b) Are A and B complements of one another?

Solution:
For $1 \leq i,\ j \leq 3$, by (i, j) we mean that Vann's card number is i, and Paul's card number is j. Clearly, the events can be written as $A = \{(1, 2), (1, 3), (2, 3)\}$, and $B = \{(2, 1), (3, 1), (3, 2)\}$.

(a) Since $A \cap B = \emptyset$, the events A and B are mutually exclusive.

(b) None of $(1, 1)$, $(2, 2)$, $(3, 3)$ belongs to $A \cup B$. Hence $A \cup B$ not being the sample space shows that A and B are not complements of one another.

Problem 1.4.31. (10 points) Among 33 students in a class, 17 of them earned A's on the midterm exam, 14 earned A's on the final exam, and 11 did not earn A's on either examination. What is the probability that a randomly selected student from this class earned an A on both exams?

Solution:
Let M and F denote the events that the randomly selected student earned an A on the midterm exam and an A on the final exam, respectively. Then

$$P(MF) = P(M) + P(F) - P(M \cup F),$$

where $P(M) = 17/33, P(F) = 14/33$, and by DeMorgan's law,

$$P(M \cup F) = 1 - P(M^c F^c) = 1 - \frac{11}{33} = \frac{22}{33}.$$

Therefore,

$$P(MF) = \frac{17}{33} + \frac{14}{33} - \frac{22}{33} = \frac{3}{11}.$$

Problem 1.7.3. (10 points) Which of the following statements are true? If a statement is true, prove it. If it is false, give a counterexample.

(a) If A is an event with probability 1, then A is the sample space.

(b) If B is an event with probability 0, then $B = \emptyset$.

Solution:

(a) False. Given a counterexample, choosing a point at random from the interval (0,1), let $A = (0, 1) - \{\frac{1}{2}\}$. A is not the sample space but $P(A) = 1$.

(b) False. A counterexample $B = \{\frac{1}{2}\}$, and $P(B) = 0$ while $B \neq \emptyset$.

Problem Ch1-Review 22. (10 points) Five customers enter a wireless corporate store to buy smartphones. If the probability that at least two of them purchase an Android smartphone is 0.6 and the probability that all of them buy non-Android smartphones is 0.17, what is the probability that exactly one Android smartphone is purchased?

Solution:
For $0 \le i \le 5$, let A_i be the event that exactly i of these customers buy an Android smartphone. We have that

$$P(A_0) + P(A_1) + \sum_{i=2}^{5} P(A_i) = 1.$$

This implies that $0.17 + P(A_1) + 0.6 = 1$. Solving this for $P(A_1)$, we obtain $P(A_1) = 0.23$.

Problem 2.2.20 (10 points) Suppose that four cards are drawn successively from an ordinary deck of 52 cards, with replacement and at random. What is the probability of drawing at least one king?

Solution:

$$1 - (\frac{12}{13})^4 = 0.274.$$

Problem 2.3.24. (10 points) Even with all the new advanced technological methods to send signals, some ships still use flags for that purpose. Each flag has its own significance as a signal, and each arrangement of two or more flags conveys a special message. If a ship has 8 flags and 4 flagpoles, how many signals can it send?

Solution:
The total number of possible signals is

$${}_8P_1 + {}_8P_2 + {}_8P_3 + {}_8P_4 = \frac{8!}{(8-1)!} + \frac{8!}{(8-2)!} + \frac{8!}{(8-3)!} + \frac{8!}{(8-4)!} = 2080.$$

Problem 2.3.28. (10 points) Five boys and five girls sit at random in a row. What is the probability that the boys are together and the girls are together?

Solution:

$$\frac{2 \times {}_5P_5 \times {}_5P_5}{{}_{10}P_{10}} = \frac{2 \times 5! \times 5!}{10!} = 0.0079.$$

Problem 2.4.50. (10 points) In a closet there are 10 pairs of shoes. If six shoes are selected at random, what is the probability of (a) no complete pairs; (b) exactly one complete pair; (c) exactly two complete pairs; (d) exactly three complete pairs?

Solution:
(a) There are $\binom{10}{6}$ combinations of 6 different pairs, and each pair has 2 different shoes. So the probability is given by

$$\frac{\binom{10}{6} \times 2^6}{\binom{20}{6}} = 0.347.$$

(b) Following the same logical reasoning as part (a), we have the probability of exactly one complete pair given by

$$\frac{\binom{10}{1} \times \binom{9}{4} \times 2^4}{\binom{20}{6}} = 0.52.$$

(c) Following the same logical reasoning as part (a), we have the probability of exactly two complete pairs given by

$$\frac{\binom{10}{2} \times \binom{8}{2} \times 2^2}{\binom{20}{6}} = 0.13.$$

(d) The probability of exactly three complete pairs is given by computing the arrangements of selecting 3 pairs from 10 pairs as follows:

$$\frac{\binom{10}{3}}{\binom{20}{6}} = 0.0031.$$

Problem Ch2-Review 12. (10 points) At a departmental party, Dr. James P. Coughlin distributes 4 baby photos of 4 of his colleagues among the guests with a list of the names of those colleagues. Each guest is invited to identify the babies in the photos. What is the probability that a participant can identify exactly 2 of the 4 photos correctly by purely random guessing?

Solution:
There are 4! ways for purely random guessing. There are $\binom{4}{2}$ ways in which 2 photos can be identified correctly by purely random guessing. For each of these choices, there is only one way not to guess the remaining pictures correctly. So the desired probability is $\frac{\binom{4}{2}}{4!} = 1/4 = 0.25$.

Problem Ch2-Review 36. (10 points) An ordinary deck of 52 cards is dealt, 13 each, at random among A, B, C, and D. What is the probability that (a) A and B together get two aces; (b) A gets all the face cards; (c) A gets five hearts and B gets the remaining eight hearts?

Solution:
(a) It is straightforward to see that the probability that A and B together get 2 aces is given by

$$\frac{\binom{4}{2} \times \binom{52-4}{24}}{\binom{52}{26}} = 0.390.$$

(b) Since A has all the 12 face cards, choose any one card from the remains. The probability for this event is given by

$$\frac{\binom{52-12}{1}}{\binom{52}{13}} = 6.299 \times 10^{-11}.$$

(c) Through the logical reasoning as in Parts (a) and (b), it can be inferred that the probability for Part (c) is

$$\frac{\binom{13}{5} \times \binom{52-13}{8} \times \binom{8}{8} \times \binom{52-13-8}{5}}{\binom{52}{13} \times \binom{39}{13}} = 0.00000261.$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)