EECS303003: Probability **Spring 2021**
College of EECS
National Tsing Hua University

Homework #5 – Solution

Coverage: chapter 8–9

Due date: 8 June, 2021

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Problem 8.1.14. (12.5 points) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{25}(x^2+y^2) & \text{if } x=1,2,\ y=0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X > Y)$, $P(X + Y \le 2)$, and $P(X + Y = 2)$.

Solution:
$P(X > Y) = p(1, 0) + p(2, 0) + p(2, 1) = 2/5$,
$P(X + Y \le 2) = p(1, 0) + p(1, 1) + p(2, 0) = 7/25$,
$P(X + Y = 2) = p(1, 1) + p(2, 0) = 6/25$.

Problem 8.2.14. (12.5 points) Let the joint probability density function of X and Y be given by

$$f(x,y) = \begin{cases} 2e^{-(x+2y)} & \text{if } x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X^2Y)$.

Solution:
Since

$$f(x,y) = e^{-x} \cdot 2e^{-2y} = f(x)f(y),$$

X and Y are independent exponential random variables with parameters 1 and 2, respectively. Therefore,

$$E(X^2Y) = E(X^2)E(Y) = 2 \cdot \frac{1}{2} = 1.$$

Problem 8.2.16. (12.5 points) Let X and Y be independent exponential random variables both with mean 1. Find $E[\max(X, Y)]$.

Solution:
Let F and f be the distribution and probability density functions of $max(X, Y)$, respectively. Clearly

$$F(t) = P[\max(X,Y) \le t] = P(X \le t,\, Y \le t) = (1-e^{-t})^2, \quad t \ge 0.$$

Thus

$$f(t) = F'(t) = 2e^{-t}(1-e^{-t}).$$

Hence

$$E[\max(X,Y)] = \int_0^\infty t \cdot 2e^{-t}(1-e^{-t})\,dt = \frac{3}{2}.$$

Problem 8.3.10. (12.5 points) First a point Y is selected at random from the interval $(0, 1)$. Then another point X is selected at random from the interval $(Y, 1)$. Find the probability density function of X.

Solution:
Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\, dy$$

Now,

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1-y} & 0 < y < 1, \quad y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $0 < x < 1$,

$$f_X(x) = \int_0^x \frac{1}{1-y}\, dy = -\ln(1-x),$$

and hence

$$f_X(x) = \begin{cases} -\ln(1-x) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 9.1.14. (12.5 points) Let $X_1, X_2, ..., X_n$ be identically distributed, independent, exponential random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that

$$E[\min(X_1, ..., X_n)] < \min\{E(X_1), ..., E(X_n)\}.$$

Solution:
The inequality follows from the fact that for $i = 1, 2, ..., n$,

$$E[\min(X1, ..., Xn)] = \frac{1}{\lambda_1 + \cdots + \lambda_n} < \frac{1}{\lambda_i} = E(X_i).$$

(See Remark 9.2.)

Problem 9.2.4. (12.5 points) Let X_1, X_2, X_3, and X_4 be independent exponential random variables, each with parameter λ. Find $P(X_{(4)} \geq 3\lambda)$.

Solution:
By Theorem 9.5, $f_4(x)$, the probability density function of $X_(4)$ is given by

$$f_4(x) = \frac{4!}{3!\,0!} \lambda e^{-\lambda x}(1 - e^{-\lambda x})^3 (e^{-\lambda x})^{4-4} = 4\lambda e^{-\lambda x}(1 - e^{-\lambda x})^3.$$

The desired probability is

$$\int_{3\lambda}^{\infty} 4\lambda e^{-\lambda x}(1 - e^{-\lambda x})^3\, dx = 1 - (1 - e^{-3\lambda^2})^4.$$

Problem 9.3.4. (12.5 points) At a certain college, 16% of the calculus students get A's, 34% B's, 34% C's, 14% D's, and 2% F's. What is the probability that, of 15 calculus students selected at random, five get B's, five C's, two D's, and at least two A's?

Solution:

$$\begin{aligned} &P(A = 2, B = 5, C = 5, D = 2, F = 1) + P(A = 3, B = 5, C = 5, D = 2, F = 0) \\ &= \frac{15!}{2!\,5!\,5!\,2!\,1!}(0.16)^2(0.34)^5(0.34)^5(0.14)^2(0.02)^1 + \frac{15!}{3!\,5!\,5!\,2!\,0!}(0.16)^3(0.34)^5(0.34)^5(0.14)^2(0.02)^0 \\ &= 0.0172. \end{aligned}$$

Problem Ch9. Review 4. (12.5 points) The joint probability density function of random variables X, Y, and Z is given by

$$f(x,y,z) = \begin{cases} c(x+y+2z) & 0 < x,\, y,\, z < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Find $P(X < 1/3 \,|\, Y < 1/2,\, Z < 1/4)$.

Solution:

(a)

$$c\int_0^1\int_0^1\int_0^1 (x+y+2x)\,dzdydx = 1 \implies c = 1/2.$$

(b)

$$\begin{aligned} P(X < 1/3 \,|\, Y < 1/2,\, Z < 1/4) &= \frac{P(X < 1/3,\, Y < 1/2,\, Z < 1/4)}{P(Y < 1/2,\, Z < 1/4)} \\ &= \frac{\int_0^{1/3}\int_0^{1/2}\int_0^{1/4} \frac{1}{2}(x+y+2z)\,dzdydx}{\int_0^{1}\int_0^{1/2}\int_0^{1/4} \frac{1}{2}(x+y+2z)\,dzdydx} \\ &= \frac{1/36}{1/8} = \frac{2}{9}. \end{aligned}$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)