EECS303003: Probability **Spring 2021**
College of EECS
National Tsing Hua University

Homework #4 – Solution
Coverage: chapter 6–7
Due date: 20 May, 2021

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Problem 6.1.12. (12.5 points) Let X denote the lifetime of a radio, in years, manufactured by a certain company. The probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{15} e^{-x/15} & \text{if } 0 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that, of eight such radios, at least four last more than 15 years?

Solution:

$P(X > 15) = \int_{15}^{\infty} \frac{1}{15} e^{-x/15} dx = \frac{1}{e}$. Thus the answer is

$$\sum_{i=4}^{8} \binom{8}{i} (\frac{1}{e})^i (1 - \frac{1}{e})^{8-i} = 0.3327.$$

Problem 6.2.8. (12.5 points) Let X be a random variable with the probability density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

(X is called a **Cauchy random variable**.) Find the probability density function of $Z = \arctan X$.

Solution:
Let G and g be the distribution and probability density functions of Z, respectively. For $-\pi/2 < z < \pi/2$,

$$\begin{aligned} G(z) &= P(\arctan X \leq z) = P(X \leq \tan z) = P(X \leq \tan z) = \int_{-\infty}^{\tan z} \frac{1}{\pi(1+x^2)} dx \\ &= \left[\frac{1}{\pi} \arctan x\right]_{-\infty}^{\tan z} = \frac{1}{\pi} z + \frac{1}{2}. \end{aligned}$$

Thus

$$g(z) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < z < \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6.3.14. (12.5 points) Suppose that X, the interarrival time between two customers entering a certain post office, satisfies

$$P(X > t) = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \quad t \geq 0,$$

where $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$, $\lambda > 0$, $\mu > 0$. Calculate the expected value of X.

Solution:
By Remark6.4,

$$E(X) = \int_0^\infty P(X > t)\,dt = \int_0^\infty (\alpha e^{-\lambda t} + \beta e^{-\mu t})\,dt = \frac{\alpha}{\lambda} + \frac{\beta}{\mu}.$$

Problem 6.3.18. (12.5 points) Let X be a continuous random variable with probability density function $f(x)$. Determine the value of y for which $E\left(|X - y|\right)$ is minimum.

Solution:
By Theorem 6.3,

$$\begin{aligned} E(|X-y|) &= \int_{-\infty}^{\infty} |x-y| f(x)\,dx = \int_{-\infty}^{y} (y-x) f(x)\,dx + \int_{y}^{\infty} (x-y) f(x)\,dx \\ &= y\int_{-\infty}^{y} f(x)\,dx - \int_{-\infty}^{y} x f(x)\,dx + \int_{y}^{\infty} x f(x)\,dx - y\int_{y}^{\infty} f(x)\,dx. \end{aligned}$$

Hence

$$\begin{aligned} \frac{dE(|X-y|)}{dy} &= \int_{-\infty}^{y} f(x)\,dx + yf(y) - yf(y) - yf(y) - \int_{y}^{\infty} f(x)\,dx + yf(y) \\ &= \int_{-\infty}^{y} f(x)\,dx - \int_{y}^{\infty} f(x)\,dx. \end{aligned}$$

Setting $\dfrac{dE(|X-y|)}{dy} = 0$, we obtain that y is the solution of the following equation:

$$\int_{-\infty}^{y} f(x)\,dx = \int_{y}^{\infty} f(x)\,dx.$$

By the definition of the median of a continuous random variable, the solution to this equation is $y = \text{median}(X)$. Hence $E(|X-y|)$ is minimum for $y = \text{median}(X)$.

Problem 7.1.10. (12.5 points) Let θ be a random number between $-\pi/2$ and $\pi/2$. Find the probability density function of $X = \tan\theta$.

Solution:
Let F be the distribution function and f be the probability density function of X. By definition,

$$\begin{aligned} F(x) &= P(X \le x) = P(\tan\theta \le x) = P(\theta \le \arctan x) \\ &= \frac{\arctan x - \left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} = \frac{1}{\pi}\arctan x + \frac{1}{2}, \quad -\infty < x < \infty. \end{aligned}$$

Thus

$$f(x) = F'(X) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Problem 7.2.14. (12.5 points) Let $X \sim N(\mu, \sigma^2)$. Prove that $P(|X-\mu| > k\sigma)$ does not depend on μ or σ.

Solution:
We have that

$$\begin{aligned} P(|X-\mu| > k\sigma) &= P(X - \mu > k\sigma) + P(X - \mu < -k\sigma) \\ &= P(Z > k) + P(Z < -k) \\ &= [1 - \Phi(k)] + [1 - \Phi(k)] = 2[1 - \Phi(k)]. \end{aligned}$$

This shows that does not depend on μ or σ.

Problem 7.3.6. (12.5 points) Let X be an exponential random variable with parameter λ. Find

$$P(|X - E(X)| \ge 2\sigma_X).$$

Solution:

$$\begin{aligned}
P\big(|X - E(X)| \geq 2\sigma_X\big) &= P\big(|X - \frac{1}{\lambda}| \geq \frac{2}{\lambda}\big) \\
&= P\big(X - \frac{1}{\lambda} \geq \frac{2}{\lambda}\big) + P\big(X - \frac{1}{\lambda} \leq -\frac{2}{\lambda}\big) \\
&= P\big(X \geq \frac{3}{\lambda}\big) + P\big(X \leq -\frac{1}{\lambda}\big) \\
&= e^{-\lambda(3/\lambda)} + 0 = e^{-3} = 0.049787.
\end{aligned}$$

Problem 7.4.4. (12.5 points) Let X be the time to failure of a mobile robot that is used as an automated guide vehicle. Suppose that $E(X) = 7$ years, and X has a gamma distribution with $r = 4$. Find $Var(X)$ and $P(5 \leq X \leq 7)$.

Solution:
Suppose that the parameters of the gamma distribution are r and λ. We are given that $r = 4$. So $E(X) = r/\lambda$ implies that $\lambda = 4/7$. Therefore, $Var(X) = r/\lambda^2 = 4/(16/49) = 49/4 = 12.25$. To find $P(5 \leq X \leq 7)$, note that the probability density function of X is

$$f(x) = \frac{(4/7)^4 x^3 e^{-4x/7}}{\Gamma(4)} = \frac{1}{3!} \cdot \big(\frac{4}{7}\big)^4 x^3 e^{-4x/7}, \quad x \geq 0.$$

Therefore,

$$\begin{aligned}
P(5 \leq X \leq 7) &= \frac{1}{6}\big(\frac{4}{7}\big)^4 \int_5^7 x^3 e^{-4x/7}\,dx \\
&= \frac{1}{6}\big(\frac{4}{7}\big)^4 \left[\frac{1}{256} e^{-4x/7}(-448x^3 - 2352x^2 - 8232x - 14406)\right]_5^7 \\
&\approx 0.246.
\end{aligned}$$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)