EECS303003: Probability **Spring 2021**
College of EECS
National Tsing Hua University

Homework #2 – Solution

Coverage: chapter 3–4

Due date: 15 April, 2021

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Problem 3.1.24. (10 points) In an international school, 60 students, of whom 15 are Korean, 20 are French, eight are Greek, and the rest are Chinese, are divided randomly into four classes of 15 each. If there is a total of eight French and six Korean students in classes A and B, what is the probability that class C has four of the remaining 12 French and three of the remaining nine Korean students?

Solution:
Reduce the sample space: 30 students of which 12 are French and nine are Korean are divided randomly into two classes of 15 each. What is the probability that one of them has exactly four French and exactly three Korean students? The solution to this problem is

$$\frac{\mathrm{C}_4^{12} \times \mathrm{C}_3^9 \times \mathrm{C}_8^9}{\mathrm{C}_{15}^{30} \times \mathrm{C}_{15}^{15}} = 0.00241.$$

Problem 3.2.4. (10 points) If eight defective and 12 nondefective items are inspected one by one, at random and without replacement, what is the probability that (a) the first four items inspected are defective; (b) from the first three items at least two are defective?

Solution:

(a) $\frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} \times \frac{5}{17} = 0.0144;$

(b) $\frac{8}{20} \times \frac{7}{19} \times \frac{12}{18} + \frac{8}{20} \times \frac{12}{19} \times \frac{7}{18} + \frac{12}{20} \times \frac{8}{19} \times \frac{7}{18} + \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} = 0.344.$

Problem 3.3.16. (10 points) A child gets lost in the Disneyland at the Epcot Center in Florida. The father of the child believes that the probability of his being lost in the east wing of the center is 0.75 and in the west wing is 0.25. The security department sends an officer to the east and an officer to the west to look for the child. If the probability that a security officer who is looking in the correct wing finds the child is 0.4, find the probability that the child is found.

Solution:
The answer is clearly 0.40. This can also be computed from

$$(0.40)(0.75) + (0.40)(0.75) = 0.40.$$

Problem 3.4.8. (10 points) Suppose that 5% of the men and 2% of the women working for a corporation make over $120,000 a year. If 30% of the employees of the corporation are women, what percent of those who make over $120,000 a year are women?

Solution:

$$\frac{(0.02)(0.30)}{(0.02)(0.30) \times (0.05)(0.70)} = 0.1463.$$

Problem 3.5.40. (10 points) A fair coin is tossed n times. Show that the events "at least two heads" and "one or two tails" are independent if $n = 3$ but dependent if $n = 4$.

Solution:
For $n = 3$, the probabilities of the given events, respectively, are

$$\mathrm{C}_2^3 \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^3 = \frac{1}{2},$$

and

$$\mathrm{C}_1^3 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \mathrm{C}_2^3 \cdot \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{3}{4}.$$

The probability of their joint occurrence is

$$\mathrm{C}_2^3 \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4}.$$

So the given events are independent. For $n = 4$, the probabilities of the given events, respectively, are

$$1 - \mathrm{C}_1^4 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 - \mathrm{C}_0^4 \cdot \left(\frac{1}{2}\right)^4 = \frac{11}{16},$$

and

$$\mathrm{C}_1^4 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 + \mathrm{C}_2^4 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = \frac{10}{16}.$$

The probability of their joint occurrence is

$$\mathrm{C}_1^4 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 + \mathrm{C}_2^4 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = \frac{10}{16} \neq \frac{11}{16} \cdot \frac{10}{16}.$$

So the given events are *not* independent.

Problem 4.2.6. (10 points) From families with three children a family is chosen at random. Let X be the number of girls in the family. Calculate and sketch the distribution function of X. Assume that in a three-child family all gender distributions are equally probable.

Solution:
Let F be the distribution function of X. Then

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/8 & 0 \le t < 1 \\ 1/2 & 1 \le t < 2 \\ 7/8 & 2 \le t < 3 \\ 1 & t \ge 3. \end{cases}$$

Problem 4.3.16. (10 points) From a drawer that contains 10 pairs of gloves, six gloves are selected randomly. Let X be the number of pairs of gloves obtained. Find the probability mass function of X.

Solution:
For $i = 0, 1, 2,$ and 3, we have

$$P(X = i) = \frac{\mathrm{C}_i^{10} \cdot \mathrm{C}_{6-2i}^{10-i} \cdot 2^{6-2i}}{\mathrm{C}_6^{20}}$$

The numerical values of these probabilities are as follows.

i	0	1	2	3
$P(X = i)$	112/323	168/323	42/323	1/323

Problem 4.4.14. (10 points) The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < -3 \\ 3/8 & -3 \le x < 0 \\ 1/2 & 0 \le x < 3 \\ 3/4 & 3 \le x < 4 \\ 1 & x \ge 4. \end{cases}$$

Calculate $E(X)$, $E(X^2 - 2|X|)$, and $E(X|X|)$.

Solution:
$p(x)$ the probability mass function of X is given by

x	-3	0	3	4
$p(x)$	$3/8$	$1/8$	$1/4$	$1/4$

Hence

$$\begin{aligned} E(X) &= -3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{5}{8}, \\ E(X^2) &= 9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{77}{8}, \\ E(|X|) &= 3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{23}{8}, \\ E(X^2 - 2|X|) &= \frac{77}{8} - 2\left(\frac{23}{8}\right) = \frac{31}{8}, \\ E(X|X|) &= -9 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8} + 9 \cdot \frac{1}{4} + 16 \cdot \frac{1}{4} = \frac{23}{8}. \end{aligned}$$

Problem 4.5.10. (10 points) In a game, Emily gives Harry three well-balanced quarters to flip. Harry will get to keep all the ones that will land heads. He will return those landing tails. However, if all three coins land tails, Harry must pay Emily two dollars. Find the expected value and the variance of Harry's net gain.

Solution:
Let X be Harry's net gain. Then

$$X = \begin{cases} -2 & \text{with probability } 1/8 \\ 0.25 & \text{with probability } 3/8 \\ 0.50 & \text{with probability } 3/8 \\ 0.75 & \text{with probability } 1/8. \end{cases}$$

Thus

$$\begin{aligned} E(X) &= -2 \cdot \frac{1}{8} + 0.25 \cdot \frac{3}{8} + 0.50 \cdot \frac{3}{8} + 0.75 \cdot \frac{1}{8} = 0.125, \\ E(X^2) &= (-2)^2 \cdot \frac{1}{8} + 0.25^2 \cdot \frac{3}{8} + 0.50^2 \cdot \frac{3}{8} + 0.75^2 \cdot \frac{1}{8} = 0.6875. \end{aligned}$$

These show that the expected value of Harry's net gain is 12.5 cents. Its variance is

$$Var(X) = 0.6875 - 0.125^2 = 0.671875.$$

Problem 4.6.1 (10 points) Mr. Norton owns two appliance stores. In store 1 the number of TV sets sold by a salesperson is, on average, 13 per week with a standard deviation of five. In store 2 the number of TV sets sold by a salesperson is, on average, seven with a standard deviation of four. Mr. Norton has a position open for a person to sell TV sets. There are two applicants. Mr. Norton asked one of them to work in store 1 and the other in store 2, each for one week. The salesperson in store 1 sold 10 sets, and the salesperson in store 2 sold six sets. Based on this information, which person should Mr. Norton hire?

Solution:
Let X_1 be the number of TV sets the salesperson in store 1 sells and X_2 be the number of TV sets the salesperson in store 2 sells. We have that $X_1^* = (10 - 13)/5 = -0.6$ and $X_2^* = (6 - 7)/4 = -0.25$. Therefore, the number of TV sets the salesperson in store 1 sells is 0.6 standard deviations below the mean, whereas the number of TV sets the salesperson in store 2 sells is 0.25 standard deviations below the mean. So Mr. Norton should hire the salesperson who worked in store 2.

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)