EECS303003: Probability **Spring 2021**
College of EECS
National Tsing Hua University

Homework #1 – Solution

Coverage: chapter 1–2

Due date: 18 March, 2021

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Problem 1.2.18. (10 points) An insurance company sells a joint life insurance policy to Alexia and her husband, Roy. Define a sample space for the death or survival of this couple in five years. What is the event that at that time only one of them lives?

Solution:
Let ad be the outcome that Alexia is dead in five years, and let al be the outcome that she lives at that time. Define rd and rl similarly. A sample space for this experiment is $S = \{(ad, rd), (ad, rl), (al, rd), (al, rl)\}$. The event that at that time only one of them lives is $E = \{(ad, rl), (al, rd)\}$.

Problem 1.4.36. (10 points) The secretary of a college has calculated that from the students who took calculus, physics, and chemistry last semester, 78% passed calculus, 80% physics, 84% chemistry, 60% calculus and physics, 65% physics and chemistry, 70% calculus and chemistry, and 55% all three. Show that these numbers are not consistent, and therefore the secretary has made a mistake.

Solution:
Let A, B, and C be the events that the student passed calculus, physics, and chemistry, respectively. By Inclution-Exclusion Principle, the probability of the student passed calculus, physics, or chemistry is $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) = 78\% + 80\% + 84\% - 60\% - 65\% - 70\% + 55\% = 102\%$. This percentage is greater than 100%, which is contradict to the Axioms of probability. Therefore, the data is incorrect.

Problem 1.7.12. (10 points) A point is selected at random from the interval $(0, 1)$. What is the probability that it is rational? What is the probability that it is irrational?

Solution:
The set of rational numbers is countable. Let $\mathbf{Q} = \{r_1, r_2, r_3, ...\}$ be the set of rational numbers in $(0, 1)$. Then

$$P(\mathbf{Q}) = P(\{r_1, r_2, r_3, ...\}) = \sum_{i=1}^{\infty} P(\{r_i\}) = 0.$$

Let $\mathbf{I}$ be the set of irrational numbers in $(0, 1)$; then

$$P(\mathbf{I}) = P(\mathbf{Q}^c) = 1 - P(\mathbf{Q}) = 1.$$

Problem Ch1-Review 30. (10 points) A number is selected at random from the set of natural numbers $\{1, 2, 3, \ldots, 1000\}$. What is the probability that it is not divisible by 4, 7, or 9?

Solution:
Let F, S, and N be the events that the number selected is divisible by 4, 7, and 9, respectively. We are interested in $P(F^c S^c N^c)$ which is equal to $1 - P(F \cup S \cup N)$ by DeMorgan's law. Now

$$\begin{aligned} P(F \cup S \cup N) &= P(F) + P(S) + P(N) - P(FS) - P(SN) - P(FN) + P(FSN) \\ &= \frac{250}{1000} + \frac{142}{1000} + \frac{111}{1000} - \frac{35}{1000} - \frac{15}{1000} - \frac{27}{1000} + \frac{3}{1000} = 0.429. \end{aligned}$$

So the desired probability is 0.571.

Problem 2.2.10 (10 points) Mr. Smith has 12 shirts, eight pairs of slacks, eight ties, and four jackets. Suppose that four shirts, three pairs of slacks, two ties, and two jackets are blue. (a) What is the probability that an all-blue outfit is the result of a random selection? (b) What is the probability that he wears at least one blue item tomorrow?

Solution:

(a) $\dfrac{4 \times 3 \times 2 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{1}{64}$.

(b) $1 - \dfrac{8 \times 5 \times 6 \times 2}{12 \times 8 \times 8 \times 4} = \dfrac{27}{32}$.

Problem 2.3.24. (10 points) Even with all the new advanced technological methods to send signals, some ships still use flags for that purpose. Each flag has its own significance as a signal, and each arrangement of two or more flags conveys a special message. If a ship has 8 flags and 4 flagpoles, how many signals can it send?

Solution:
The total number of possible signals is

$$\mathrm{P}_1^8 + \mathrm{P}_2^8 + \mathrm{P}_3^8 + \mathrm{P}_4^8 = \frac{8!}{(8-1)!} + \frac{8!}{(8-2)!} + \frac{8!}{(8-3)!} + \frac{8!}{(8-4)!} = 2080.$$

Problem 2.3.28. (10 points) Five boys and five girls sit at random in a row. What is the probability that the boys are together and the girls are together?

Solution:

$$\frac{2 \times \mathrm{P}_5^5 \times \mathrm{P}_5^5}{\mathrm{P}_{10}^{10}} = \frac{2 \times 5! \times 5!}{10!} = 0.0079.$$

Problem 2.4.50. (10 points) In a closet there are 10 pairs of shoes. If six shoes are selected at random, what is the probability of (a) no complete pairs; (b) exactly one complete pair; (c) exactly two complete pairs; (d) exactly three complete pairs?

Solution:

(a) $\dfrac{\mathrm{C}_6^{10} \times 2^6}{\mathrm{C}_6^{20}} = 0.347;$

(b) $\dfrac{\mathrm{C}_1^{10} \times \mathrm{C}_4^{9} \times 2^4}{\mathrm{C}_6^{20}} = 0.520;$

(c) $\dfrac{\mathrm{C}_2^{10} \times \mathrm{C}_2^{8} \times 2^2}{\mathrm{C}_6^{20}} = 0.130;$

(d) $\dfrac{\mathrm{C}_3^{10}}{\mathrm{C}_6^{20}} = 0.0031.$

Problem Ch2-Review 12. (10 points) At a departmental party, Dr. James P. Coughlin distributes 4 baby photos of 4 of his colleagues among the guests with a list of the names of those colleagues. Each guest is invited to identify the babies in the photos. What is the probability that a participant can identify exactly 2 of the 4 photos correctly by purely random guessing?

Solution:
There are 4! ways for purely random guessing. There are C_2^4 ways in which 2 photos can be identified correctly by purely random guessing. For each of these choices, there is only one way not to guess the remaining pictures correctly. So the desired probability is $\dfrac{\mathrm{C}_2^4}{4!} = 1/4 = 0.25$.

Problem Ch2-Review 36. (10 points) An ordinary deck of 52 cards is dealt, 13 each, at random among A, B, C, and D. What is the probability that (a) A and B together get two aces; (b) A gets all the face cards; (c) A gets five hearts and B gets the remaining eight hearts?

Solution:

(a) $\dfrac{\mathrm{C}_2^4 \times \mathrm{C}_{24}^{48}}{\mathrm{C}_{26}^{52}} = 0.390;$

(b) $\dfrac{\mathrm{C}_1^{40}}{\mathrm{C}_{13}^{52}} = 6.299 \times 10^{-11};$

(c) $\dfrac{\mathrm{C}_5^{13} \times \mathrm{C}_8^{39} \times \mathrm{C}_8^8 \times \mathrm{C}_5^{31}}{\mathrm{C}_{13}^{52} \times \mathrm{C}_{13}^{39}} = 0.00000261.$

References

[1] Saeed Ghahramani, *Fundamentals of Probability: With Stochastic Processes,* Chapman and Hall/CRC; 4th edition (September 4, 2018)