EECS303003: Probability **Spring 2021**
College of EECS
National Tsing Hua University

Final Exam
Coverage: Chapter 6–11
Date: June 22
Time: 13:05 – 15:15

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1. **Notice**:
   (i) There are 9 questions with total 110 points.
   (ii) You need to provide clear logical reasoning for all your answers.
2. **Preparations**:
   (i) Enough answer sheets(A4 size) and pens.
   (ii) Join the course group (Probability_10920EECS) on Skype.
   (iii) Electronic devices that can upload the file of your answer sheets to iLMS. Also, be familiar with how to upload it.
3. **How the exam proceeds**:
   (i) Please join the skype group call at 13:10 (10 minutes before the exam).
   (ii) Turn on your webcam and shoot yourself until the end of the exam. TA may ask you to adjust the angle of the webcam.
   (iii) The file of examination questions will be posted on the iLMS platform at 12:55, 6/22 (Tue), for you to download and answer by handwriting.
   (iv) Take photos of your handwritten answer sheets (the photo must be clear and complete), and upload the photo file (a single compressed file) of your answer sheets before 15:20 to iLMS platform. Late uploading will be counted as 0 point.
   (v) The file name of your answer sheets should follow the order: student ID_Name_page number, for example: 107064xxx_Wang Xiaw Ming_01.rar(or .zip). Otherwise, your examination grades may be decreased by some points.
4. **Remarks**:
   (i) During the exam, it is allowed to refer to the textbook. Those who do not have the textbook can prepare paper notes for the exam. It is forbidden to use electronic devices for on-line information search or discussion.
   (ii) Please make sure that the network connection of the electronic device is reliable with enough power.
   (iii) Blurry or incomplete uploaded photos may lead to negative effects (e.g., some points loss) on the grading.
   (iv) Please autograph on the first-page of your answer sheet.
   (v) Make sure to know how to upload files by testing iLMS in advance. We cannot help out when uploading or downloading failure occurs during the examination.

Problem 1. (10 points) U is a continuous uniform random variable such that $E(U) = 10$ and $P(U > 12) = 1/4$. What is $P(U < 9)$?
Solution:
Since U is a continuous uniform random variable with $E(U) = 10$, we know that $u = 10$ is the midpoint of (a, b) of a uniform (a, b) random variable. That is, for some constant $c > 0$, U is a continuous uniform $(10-c, 10+c)$ random variable with probability density function(PDF)

$$f(U = u) = \begin{cases} 1/2c & 10 - c < u < 10 + c \\ 0 & \text{otherwise.} \end{cases}$$

This implies

$$\begin{aligned} \frac{1}{4} = P(U > 12) &= \int_{12}^{\infty} f(U = u)\, du \\ &= \int_{12}^{10+c} \frac{1}{2c}\, du = \frac{(10 + c) - 12}{2c} = \frac{1}{2} - \frac{1}{c}. \end{aligned}$$

This implies $1/c = 1/4$ or $c = 4$. Thus U is a uniform random variable with PDF

$$f(U = u) = \begin{cases} 1/8 & 6 < u < 14 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$P(U < 9) = \int_{-\infty}^{9} f(U = u)\, du = \int_{6}^{9} \frac{1}{8}\, du = \frac{3}{8}.$$

Problem 2. (10 points)

(a) Show that an exponential distribution is memoryless.

(b) Assuming the service time (in minutes) is an exponential distribution with parameter $\lambda = 1/10$. A customer Y arrives when customer X has been served for 5 minutes. On average, how long does the customer Y need to wait for customer X to complete the service?

Solution:

(a) The cdf of the exponential random variable with parameter λ is given by $P(X < t) = 1 - e^{-\lambda t},\ t > 0$. For all $s, t > 0$

$$\begin{aligned} P(X > s + t \,|\, X > t) &= \frac{P(X > s + t,\, X > t)}{P(X > t)} \\ &= \frac{P(X > s + t)}{P(X > t)} \\ &= \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda t})} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} = P(X > s). \end{aligned}$$

Hence X is memoryless.

(b) Since the service distribution is memoryless, the distribution of the time required for one completion is $1/\lambda = 10$ minutes.

Problem 3. (15 points) Random variables X and Y have the joint PDF

$$f(x, y) = \begin{cases} c & x \geq 0,\ y \geq 0,\ x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P(X \leq Y)$?

(c) What is $P(X + Y \leq 1/2)$?

Solution:

(a) To find the constant c we integrate over the domain of $f(x, y)$. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = \left[cx - \frac{cx^2}{2} \right]_0^1 = \frac{c}{2} = 1.$$

Therefore $c = 2$.

(b) To find the $P(X \leq Y)$ we integrate over the area $\{x \geq 0,\ y \geq 0,\ x \leq y \leq 1 - x\}$.

$$\begin{aligned} P(X \leq Y) &= \int_0^{1/2} \int_x^{1-x} 2\, dy\, dx \\ &= \int_0^{1/2} (2 - 4x)\, dx \\ &= 1/2. \end{aligned}$$

(c) To find the $P(X + Y \leq 1/2)$ we integrate over the area $\{x \geq 0,\ y \geq 0,\ x + y \leq 1/2\}$.

$$\begin{aligned} P(X + Y \leq 1/2) &= \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \\ &= \int_0^{1/2} (1 - 2x)\, dx \\ &= \left[x - x^2 \right]_0^{1/2} = \frac{1}{4}. \end{aligned}$$

Problem 4. (10 points) Let X be the time to failure of a mobile robot that is used as an automated guide vehicle. Suppose that $E(X) = 1$ years, and X has a gamma distribution with $r = 2$ and rate λ. Find $Var(X)$ and $P(0 \leq X \leq 1)$.

Solution:
Suppose that the parameters of the gamma distribution are r and λ. We are given that $r = 2$. So $E(X) = r/\lambda$ implies that $\lambda = 2/1 = 2$. Therefore, $Var(X) = r/\lambda^2 = 2/4 = 1/2$. To find $P(0 \le X \le 1)$, note that the probability density function of X is

$$f(x) = \frac{2(2x)^1 e^{-2x}}{\Gamma(2)} = \frac{1}{1!} \cdot 4xe^{-2x} = 4xe^{-2x}, \quad x \ge 0.$$

Therefore,

$$\begin{aligned}
P(0 \le X \le 1) &= 4\int_0^1 xe^{-2x}\,dx = 4\int_0^1 \frac{x}{-2}\,de^{-2x} = -2\int_0^1 x\,de^{-2x} \\
&= -2\left[xe^{-2x}\Big|_0^1 - \int_0^1 e^{-2x}\,dx \right] \\
&= -2\left[(e^{-2} - 0) - \frac{e^{-2x}}{-2}\Big|_0^1 \right] \\
&= -2\left[e^{-2} + \frac{1}{2}(e^{-2} - 1) \right] \\
&= -2\left[-\frac{1}{2} + \frac{3}{2}e^{-2} \right] = 1 - \frac{3}{e^2}.
\end{aligned}$$

Problem 5. (10 points) X and Z are independent random variables with $E(X) = E(Z) = 0$ and variance $Var(X) = 1$ and $Var(Z) = 16$. Let $Y = X + Z$. Find the correlation coefficient ρ of X and Y. Are X and Y independent?
Solution:
Independence of X and Z implies

$$Var(Y) = Var(X) + Var(Z) = 1 + 16 = 17.$$

Since $E(X) = E(Y) = 0$, the covariance of X and Y is

$$Cov(X, Y) = E(XY) - 0 = E[X(X + Z)] = E(X^2) + E(XZ).$$

Since X and Z are independent and $E(X) = 0$, we have

$$E(XZ) = E(X)E(Z) = 0$$

and

$$E(X^2) = Var(X) = 1.$$

Finally, the correlation coefficient is

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{1}{\sqrt{17}}.$$

Since $\rho_{X,Y} \ne 0$, we conclude that X and Y are not independent.

Problem 6. (15 points) X_1, X_2, X_3 are independent and identical exponential(λ) random variables. Find the PDF of $V = \min(X_1, X_2, X_3)$.
Solution:
We first derive the CDF of V. Since each X_i is nonnegative, V is nonnegative, thus $F(V = v) = 0$ for $v < 0$. For $v \geq 0$, independence of the X_i yields

$$\begin{aligned} F(V = v) = P(V \leq v) &= P\left[\min(X_1, X_2, X_3) \leq v\right] \\ &= 1 - P\left[\min(X_1, X_2, X_3) > v\right] \\ &= 1 - P\left(X_1 > v, X_2 > v, X_3 > v\right) \\ &= 1 - P(X_1 > v)P(X_2 > v)P(X_3 > v). \end{aligned}$$

Note that independence of the X_i was used in the final step. Since each X_i is an exponential(λ) random variable, for $v \geq 0$,

$$P(X_i > v) = 1 - F(X_i = v) = e^{-\lambda v}.$$

Thus,

$$F(V = v) = 1 - (e^{-\lambda v})^3 = 1 - e^{-3\lambda v}.$$

The complete expression for the CDF of V is

$$F(V = v) = \begin{cases} 0 & v < 0 \\ 1 - e^{-3\lambda v} & v \geq 0. \end{cases}$$

By taking the derivative of the CDF, we obtain the PDF

$$f(V = v) = \begin{cases} 0 & v < 0 \\ 3\lambda e^{-3\lambda v} & v \geq 0. \end{cases}$$

We see that V is an exponential random variable with parameter 3λ.

Problem 7. (15 points) X and Y are independent random variables with PDFs

$$f(X = x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f(Y = y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{X > Y\}$.

(a) What are $E(X)$ and $E(Y)$?

(b) What is $E(X|A)$?

Solution:

(a) We can calculate the unconditional expectations, $E(X)$ and $E(Y)$, using the marginal PDFs $f(X = x)$ and $f(Y = y)$.

$$E(X) = \int_{-\infty}^{\infty} x f(X = x)\, dx = \int_0^1 2x^2\, dx = 2/3$$
$$E(Y) = \int_{-\infty}^{\infty} y f(Y = y)\, dy = \int_0^1 3y^3\, dy = 3/4$$

(b) We will do this using the conditional joint PDF $f_{X,Y|A}(x, y)$ The first step is to write down the joint PDF of X and Y:

$$f(x, y) = f(x)f(y) = \begin{cases} 6xy^2 & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The event A has probability

$$\begin{aligned} P(A) &= \iint_A f(x, y)\, dx\, dy = \iint_{x>y} 6xy^2\, dx\, dy \\ &= \int_0^1 \int_0^x 6xy^2\, dy\, dx = \int_0^1 2x^4\, dx = 2/5. \end{aligned}$$

The conditional joint PDF of X and Y given A is

$$\begin{aligned} f_{X,Y|A}(x, y) &= \begin{cases} \dfrac{f(x, y)}{P(A)} & (x, y) \in A \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 15xy^2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the conditional expected value of X given A is

$$\begin{aligned} E(X|A) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y|A}(x, y)\, dx\, dy \\ &= \int_0^1 \int_0^x 15xy^2\, dy\, dx \\ &= \int_0^1 5x^5\, dx = 5/6. \end{aligned}$$

Problem 8. (15 points) We start with a stick of length l. We break it at a point which is chosen randomly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the stick that we were left with. What is the expected length of the stick that we are left with, after breaking twice?
Solution:
Let Y be the length of the stick after we break for the first time and X be the length after the second time. Then

$$f_Y(y) = \begin{cases} 1/l & 0 \le y \le l \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} 1/y & 0 \le x \le y \\ 0 & \text{otherwise.} \end{cases}$$

By the law of iterated expectations, we have

$$\begin{aligned} E(X) &= E[E(X|Y)] \\ &= \int_0^l E(X|Y=y) f_Y(y)\,dy \\ &= \int_0^l \left[\int_0^y x f_{X|Y}(x|y)\,dx\right] f_Y(y)\,dy \\ &= \int_0^l \left[\int_0^y x \cdot \frac{1}{y}\,dx\right] \cdot \frac{1}{l}\,dy \\ &= \int_0^l \frac{y}{2} \cdot \frac{1}{l}\,dy = \frac{l}{4}. \end{aligned}$$

Problem 9. (10 points) Which of the following statements are TRUE and which are FALSE? Justify your answers by clear logical reasoning.

(a) If X is an exponential random variable with mean 1, then $P(X \ge a) \le \frac{1}{a}$, $\forall a > 0$.

(b) If X is a standard normal random variable, then $P(X^2 + 2X + 1 \ge a) \le \frac{3}{a}$, $\forall a > 0$.

(c) If X is normal random variable such that $E(X) = 1$ and $E(X^2) = 4$, then $P(|X - 1| \ge 2) \le \frac{3}{4}$.

(d) Sum of two independent standard normal random variables is also standard normal random variable (The probability density function $f(x)$ and moment generating function $M_X(t)$ of a standard normal random variable X are given by $(1/\sqrt{2\pi})e^{(-x^2/2)}$ and $e^{(t^2/2)}$, respectively).

Solution:

(a) TRUE. According to the Mackov's inequality,

$$P(X \ge a) \le \frac{E(X)}{a} = \frac{1/1}{a}, \ \forall a > 0.$$

(b) TRUE. According to the Mackov's inequality,

$$P(X^2+2X+1\geq a)\leq \frac{E(X^2+2X+1)}{a}=\frac{E(X^2)+2E(X)+1}{a}=\frac{1+0+1}{a}=\frac{2}{a}\leq\frac{3}{a},\ \forall a>0.$$

Note that $E(X^2)=Var(X)-(E(X))^2=1-0=1$.

(c) TRUE. According to Chebyshev's inequality,

$$P(|X-1|\geq 2)\leq \frac{E[(X-1)^2]}{2^2}=\frac{Var(X)}{4}=\frac{4-1^2}{4}=\frac{3}{4}.$$

(d) FALSE. Let $X_1\sim N(0,1)$ and $X_2\sim N(0,1)$ be independent and $Y=X_1+X_2$. We have

$$M_Y(t)=M_{X_1}(t)M_{X_2}(t)=e^{(t^2/2)}e^{(t^2/2)}=e^{t^2}=M_{X_1}(\sqrt{2}t)=M_{\sqrt{2}X_1}(t).$$

This implies Y and $\sqrt{2}X_1$ have the same pdf, namely, $Y\sim N(0,2)$.