國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 9: discrete random variable

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Quick review

PMF:

$$p_X(x) = P(X = x)$$

Joint PMF:

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

Conditional PMF:

$$p_{X|Y}(x|y) = P(X = x|Y = y)$$

Marginal PMF:

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

Joint PMF as conditional PMF:

$$p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y|x)$$

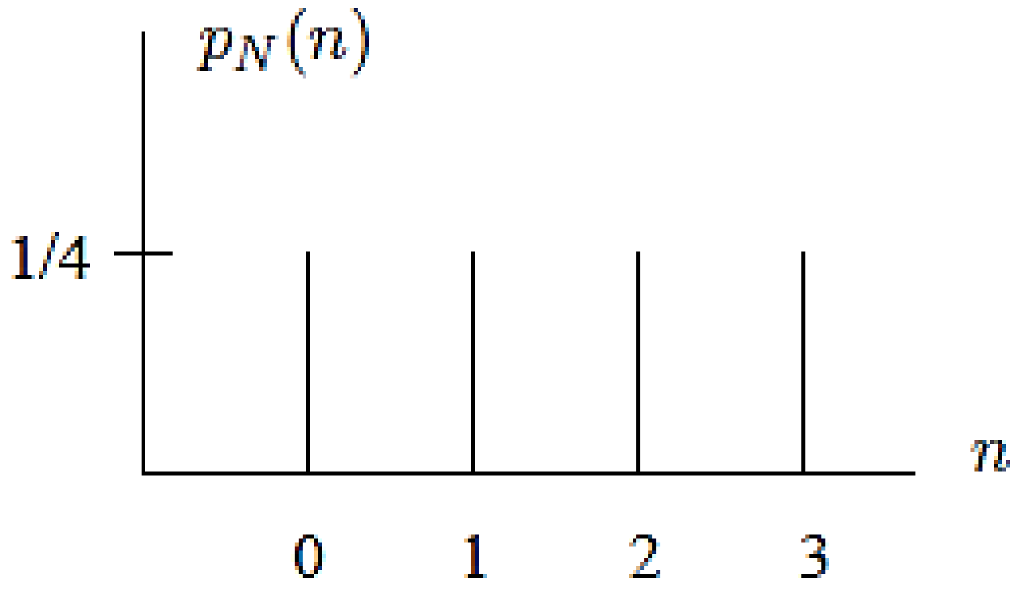
- Expectation
  - E[X]
  - Average, values taken by variable X weighted by probability
- Variance
  - Distance (spread) from the average
  - $E[X^2] - (E[X])^2$
- These functions (operator) can be applied onto unconditional, joint, or conditional probability (note*: they are all the same, just probability with different sample space)

p1

Consider an experiment in which a fair four-sided (with faces labeled 0,1,2,3) is thrown once to determine how many times a fair coin is to flipped. In the sample space of this experiment, random variables N and K are defined by:

- N = the result of die roll
- K = the total number of heads resulting from the coin flips

a) Determine and sketch $p_N(n)$



b) Determine and tabulate $p_{N,K}(n,k)$

When N = 0, the coin is not flipped at all, so K=0. When N = n for n $\in$ {1, 2, 3}, the coin is flipped n times, resulting in K with a distribution that is conditionally binomial. The binomial probabilities are all multiplied by ¼ because $p_N(n)$ = ¼ for n $\in$ {1, 2, 3},

$$p_{K|N}(k|n) \sim B(n,k)$$

To construct the following table

$$p_{K|N}(k|n)p_N(n) = p_{K,N}(k,n)$$

	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$	$1/4$	0	0	0
$n=1$	$1/8$	$1/8$	0	0
$n=2$	$1/16$	$1/8$	$1/16$	0
$n=3$	$1/32$	$3/32$	$3/32$	$1/32$

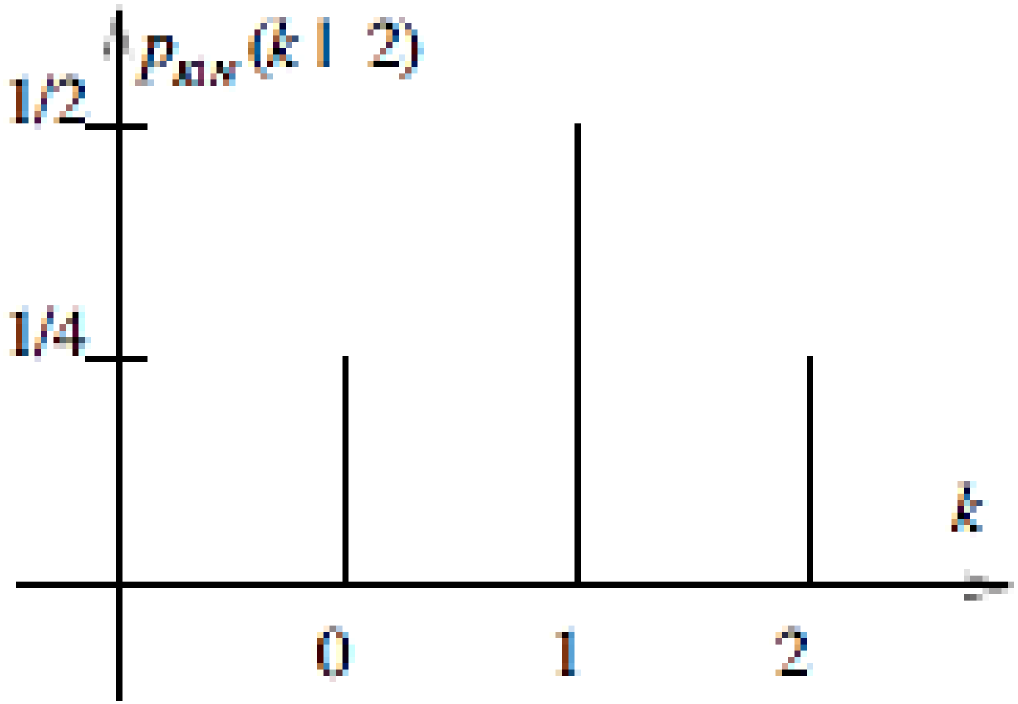
c) Determine and sketch $p_{K|N}(k|2)$

Conditioning on N =2, K is binomial random variable. So we immediately see that

$$p_{K|N}(k|2) = \begin{cases} \frac{1}{4}, \text{if k=0} \\ \frac{1}{2}, \text{if k=1} \\ \frac{1}{4}, \text{if k=2} \end{cases}$$

	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$	1/4	0	0	0
$n=1$	1/8	1/8	0	0
$n=2$	1/16	1/8	1/16	0
$n=3$	1/32	3/32	3/32	1/32

This is essentially a normalized row of the table on the left
Or you can apply binomial PMF


pX|N(k | 2)
1/2
1/4
k
0
1
2

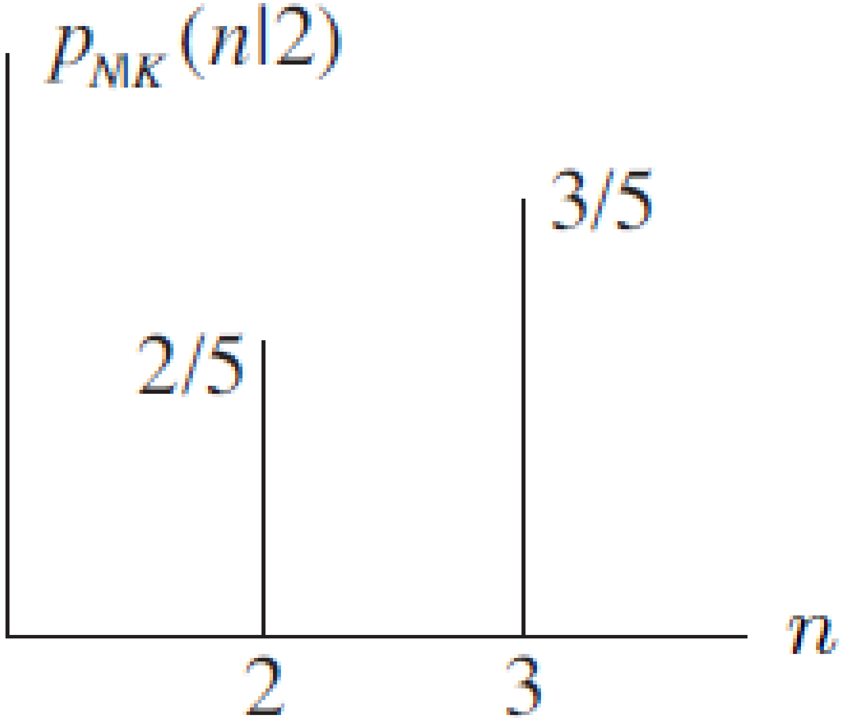
d) Determine and sketch $p_{N|K}(n|2)$

To get K=2, there must been at least 2 coin tosses, so only N=2, and N=3

$$p_{N|K}(2|2) = \frac{P(\{N=2\} \cap \{K=2\})}{P(\{K=2\})} = \frac{\frac{1}{16}}{\frac{1}{16}+\frac{3}{32}} = \frac{2}{5}$$

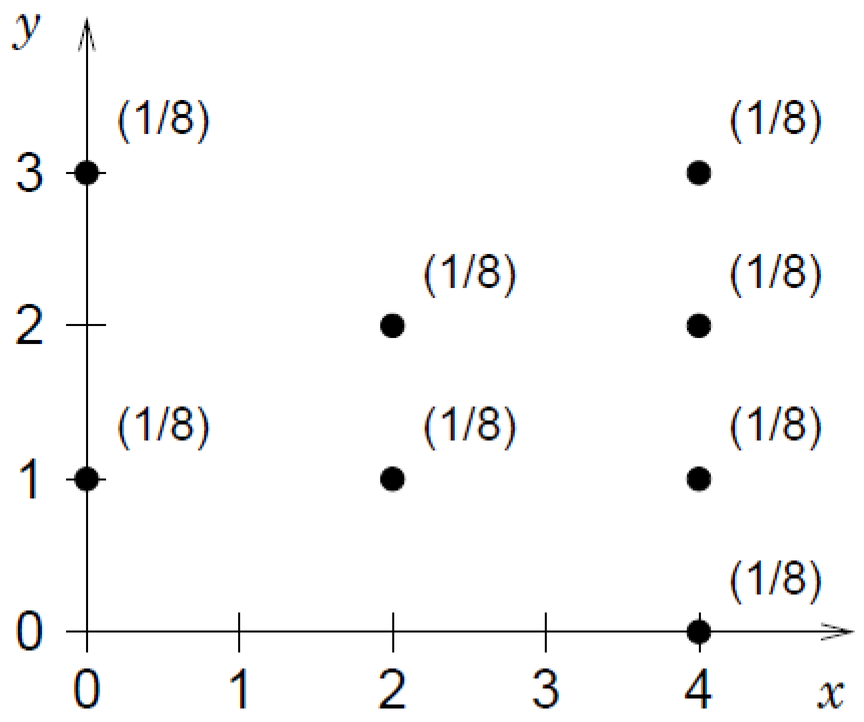
$$p_{N|K}(3|2) = \frac{P(\{N=3\} \cap \{K=2\})}{P(\{K=2\})} = \frac{\frac{3}{32}}{\frac{1}{16}+\frac{3}{32}} = \frac{3}{5}$$

	$k=0$	$k=1$	$k=2$	$k=3$
$n=0$	$1/4$	0	0	0
$n=1$	$1/8$	$1/8$	0	0
$n=2$	$1/16$	$1/8$	$1/16$	0
$n=3$	$1/32$	$3/32$	$3/32$	$1/32$


$p_{MK}(n|2)$
3/5
2/5
n
2
3

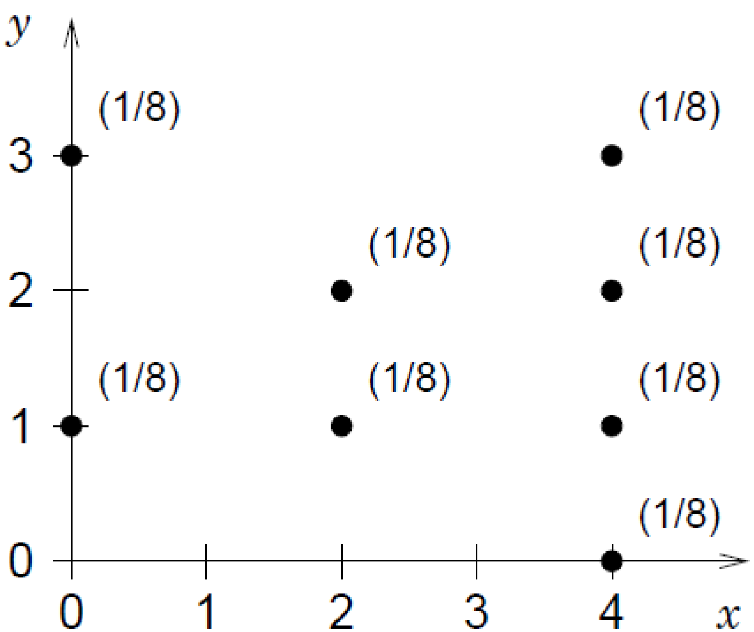
p2

Consider an outcome space comprising eight equally-likely event points as show below:



a) Which value(s) of x maximize E[Y|X=x]?

$$E[Y|X=0] = \frac{1}{2} * 3 + \frac{1}{2} * 1 = 2$$
$$E[Y|X=2] = \frac{1}{2} * 2 + \frac{1}{2} * 1 = 1.5$$
$$E[Y|X=4] = \frac{1}{4} * 3 + \frac{1}{4} * 2 + \frac{1}{4} * 1 = 1.5$$



X = 0 maximizes that E[Y|X=x]

b) Which values of y maximize var(X|Y=y)?

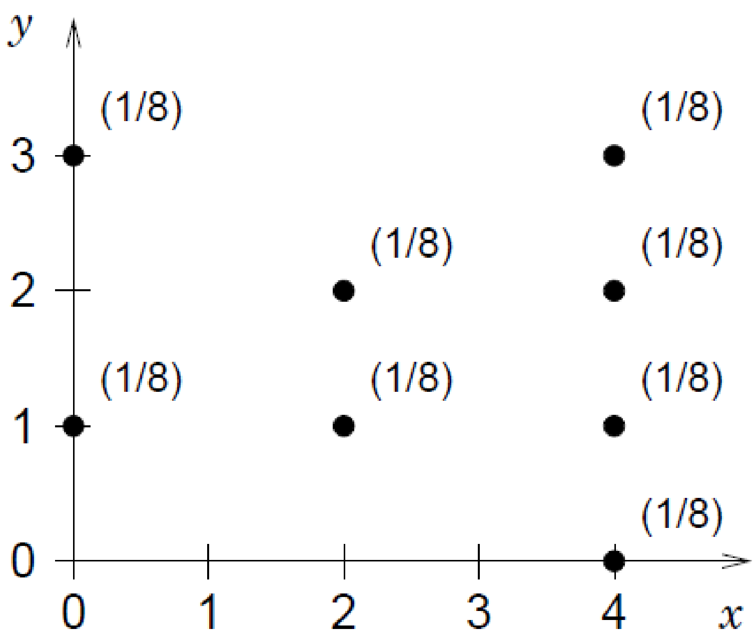
$$var(X|Y=0) = ?$$

$E[X|Y=0] = 4$, variance of a constant = 0

$$E[X|Y=1] = 2, var(X|Y=1) = E_{X|Y=1}[(X-2)^2]$$

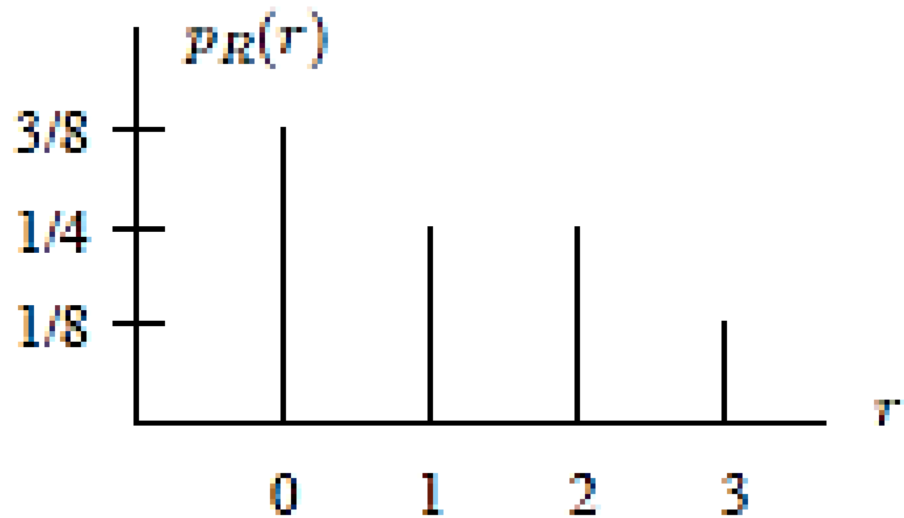
$$= \frac{1}{3}*4 + \frac{1}{3}*0 + \frac{1}{3}*4 = 8/3$$

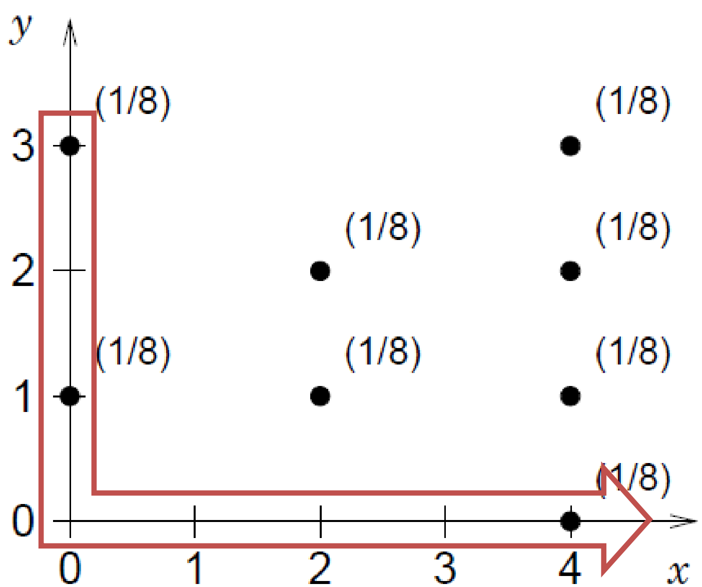
REPEAT



$$var(X|Y=y) = \begin{cases} 0, \text{if } y=0 \\ \frac{8}{3}, \text{if y=1} \\ 1, \text{if y=2} \\ 4, \text{if y=3} \\ \text{undefined,} \end{cases}$$

c) Let R = min(X,Y). Prepare a neat fully labeled sketch of $p_R(r)$



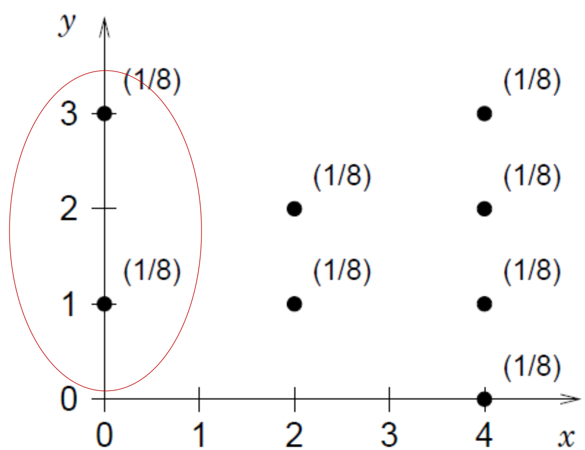


A total of 8 points
NOW GO THROUGH EACH CASE TO ADD

d) Let A denote the events $X^2 \geq Y$. Determine numerical values for the quantities of E[XY] and E[XY|A]

$$E[XY] = \sum_{x,y} (XY) p_{X,Y}(x,y)$$
$$= 1/8(0*3+4*3+2*2+4*2+0*1+2*1+4*1+4*0$$
$$= 15/4$$

Conditioning on A removes the point masses at (0,1) (0,3). The conditional probability of each of the remaining point masses is thus 1/6



$$E[XY|A] = \frac{1}{6}(4*3+2*2+4*2+2*1+4*1+4*0) = 5$$

p3

Variance of geometric distribution?

Back to geometric case

- A1: {X=1}, A2:{X>1}

$$E[X^2 | X = 1] = 1 * \mathrm{p}$$
$$E[X^2 | X > 1] = E[(1 + X)^2] = 1 + 2E[X] + E[X^2]$$

$$E[X^2] = p * 1 + (1 - p) * \left(1 + 2 * \frac{1}{p} + E[X^2]\right)$$

TOTAL EXPENCTATION THEOREM (SEPARATE INTO TWO CASES)

$$E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

$$var(X) = E[X^2] - E[X]^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1 - p}{p^2}$$

p5

Consider a sequence of independent tosses of a biased coin at times t = 0,1,2,… On each toss, the probability of a ‘head’ is p, and the probability of a ‘tail’ is 1-p. A reward of one unit is given each time that a ‘tail’ follows immediately after ‘head’. Let R be the total reward paid in times 1,2,…,n. Find E[R] and var(R)

Let I_k be the reward paid at time k, we have
$E[I_k] = P(I_k = 1) = P(T \text{ at time } k \text{ and } H \text{ at time } k-1) * 1$
$= p(1-p)$

Computing E[R] is immediate because:

$$E[R] = E\left[\sum_{k=1}^{n} I_k\right] = \sum_{k=1}^{n} E[I_k] = np(1-p)$$

The variance is not as easy because I_ks are not all independent

$$E\left[I_k^2\right] = p(1-p) * 1^2$$

$$E[I_k I_{k+1}] = 0 \text{ because reward can only happen at a time!}$$

$$E[I_k I_{k+l}] = E[I_k]E[I_{k+l}] = p^2(1-p)^2, \text{for l} \geq 2$$

$$E[R^2] = E\left[\left(\sum_{k=1}^{n} I_k\right)\left(\sum_{m=1}^{n} I_m\right)\right] = \sum_{k=1}^{n}\sum_{m=1}^{n} E[I_k I_m]$$

- When k = m, the summation is ~~n~~$p(1-p)$
  - There are n terms of this

$$E\left[I_k^2\right] = p(1-p) * 1^2$$
$$E[I_k I_{k+1}] = 0$$
$$E[I_k I_{k+l}] = p^2(1-p)^2,$$

- When $|k-m| = 1$, summation is 0
  - There are 2n-2 terms of this kind
- So the rest has summation of $(n^2 - 3n + 2) * (p^2 * (1-p)^2)$
- Put it together:

$$Var[R^2] = E[R^2] - (E[R])^2$$
$$= np(1-p) + \left(n^2 - 3n + 2\right)p^2(1-p)^2 - n^2p^2(1-p)^2$$
$$= np(1-p) - (3n-2)p^2(1-p)^2$$

p6

The joint PMF of the random variable X and Y is given by the following table:

$y = 3$	c	c	$2c$
$y = 2$	$2c$	0	$4c$
$y = 1$	$3c$	c	$6c$
	$x = 1$	$x = 2$	$x = 3$

a) Find the value of the constant c:

We can find c knowing that the probability of the entire sample space must equal to 1

$$1 = \sum_{x=1}^{3}\sum_{y=1}^{3} p_{X,Y}(x,y) = c + c + 2c + 2c + 4c + 3c + c + 6c$$
$$= 20\mathrm{c}$$

$$c = \frac{1}{20}$$

b) Find $p_Y(2)$

$y=3$	c	c	$2c$
$y=2$	$2c$	0	$4c$
$y=1$	$3c$	c	$6c$
	$x=1$	$x=2$	$x=3$

$$p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x,2) = 2c + 0 + 4c = \frac{3}{10}$$

c) Consider the random variable $Z = YX^2$. Find the $E[Z|Y=2]$

$$E[Z|Y=2] = E[YX^2|Y=2] = E[2X^2|Y=2] = 2E[X^2|Y=2]$$

$$p_{X|Y}(x|2) = \frac{p_{X,Y}(x,2)}{p_Y(2)}$$

$y=3$	c	c	$2c$
$y=2$	$2c$	0	$4c$
$y=1$	$3c$	c	$6c$
	$x=1$	$x=2$	$x=3$

Therefore,

$$p_{X|Y}(x|2) = \begin{cases} \frac{1}{3}, if\ x = 1 \\ \frac{2}{3}, if\ x = 3 \\ 0, otherwise \end{cases}$$

$$c = \frac{1}{20}$$

$$E[Z|Y = 2] = 2\sum_{x=1}^{3} x^2 p_{X|Y}(x|2) = 2\left(1^2 * \frac{1}{3} + 3^2 * \frac{2}{3}\right)$$

$$= \frac{38}{3}$$

d)

Conditioned on the event that $X \neq 2$, are X and Y independent? Give a one line justification.

$y = 3$	c	c	$2c$
$y = 2$	$2c$	0	$4c$
$y = 1$	$3c$	c	$6c$
	$x = 1$	$x = 2$	$x = 3$

Yes, let's look at the following:

$$P(X = x|Y = y, X \neq 2) = P(X = x|X \neq 2)$$

think of an case:

$$P(X = 1|Y = 1, X \neq 2) = P(X = 1|Y = 3, X \neq 2)$$
$$= P(X = 1|X \neq 2) = 1/3$$

Given that $X \neq 2$, the distribution of X is the same given Y=y

e) Find the conditional variance of Y given that X = 2

$$p_{Y|X}(y|x=2) = \frac{p_{XY}(2,y)}{p_X(2)}$$

$$p_X(2) = \sum_{y=1}^{3} p_{XY}(2,y) = c + 0 + c = \frac{1}{10}$$

$y=3$	c	c	$2c$
$y=2$	$2c$	0	$4c$
$y=1$	$3c$	c	$6c$
	$x=1$	$x=2$	$x=3$

$$p_{Y|X}(y|2) = \begin{cases} \dfrac{1/20}{1/10} = \dfrac{1}{2}, if\ y = 1 \\ \dfrac{1/20}{1/10} = \dfrac{1}{2}, if\ y = 3 \\ 0 \end{cases}$$

- Conditional variance? Same thing as plain variance

$$var(Y|X = 2) = E[Y^2|X = 2] - E[Y|X = 2]^2$$

Just need to compute individual term:

$$E[Y^2|X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(x|y = 2)$$

$$= (1^2) * \frac{1}{2} + (3^2) * \frac{1}{2} = 5$$

$$E[Y|X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y|2) = 1 * \frac{1}{2} + 3 * \frac{1}{2} = 2$$

$$var(Y|X = 2) = 5 - 4 = 1$$

p7

- Suppose that X and Y are independent, identically distributed (iid), geometric random variables with parameter p, we want to show the following:

- $P(X = i|X + Y = n) = \frac{1}{n-1}, \text{for } i = 1,2, \dots, n - 1$

$$P(X = i|X + Y = n) = \frac{P(\{X = i\} \cap \{X + Y = n)}{P(X + Y = n)}$$

The event $\{X = i\} \cap \{X + Y = n\}$ in the numerator in equivalent to $\{X = i\} \cap \{Y = n - i\}$, taking this in combination with the fact hat X and Y are independent

$$P(X + Y = n) = \sum_{i=1}^{n-1} P(X = i)P(X + Y = n|X = i) =$$

$$\sum_{i=1}^{n-1} P(X = i)P(i + Y = n|X = i) =$$

$$\sum_{i=1}^{n} P(X = i)P(Y = n - i|X = i)$$

$$= \sum_{i=1}^{n-1} P(X = i)P(Y = n - i)$$

Total probability theorem

We only get non-zero probability for i=1, …, n-1 since X and Y are both geometric random variables

So now we can write it completely from the previous slides:

$$P(X = i | X + Y = n) = \frac{P(X = i)P(Y = n - i)}{\sum_{i=1}^{n-1} P(X = i)P(Y = n - i)}$$

$$= \frac{(1-p)^{i-1}p(1-p)^{n-i-1}p}{\sum_{i=1}^{n-1}(1-p)^{i-1}p(1-p)^{n-i-1}p} = \frac{(1-p)^n}{\sum_{i=1}^{n-1}(1-p)^n}$$

$$= \frac{(1-p)^n}{(1-p)^n\sum_{i=1}^{n-1} 1} = \frac{1}{n-1}$$

P8

- A simple example of a random variable is the indictor of an event A, which is denoted by I_A:

$$I_A(w) = \begin{cases} 1, \text{if } w \in A \\ 0, \text{otherwise} \end{cases}$$

a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A, I_B are independent

We know that I_A is a random variable that maps a 1 to the real number if w occurs within an event A, and maps a 0 to the real number line if w occurs outside the event A. A similar argument holds for event B, so we have the following:

$$I_A(\omega) = \begin{cases} 1, & \text{with probability } \mathbf{P}(A) \\ 0, & \text{with probability } 1 - \mathbf{P}(A) \end{cases}$$

$$I_B(\omega) = \begin{cases} 1, & \text{with probability } \mathbf{P}(B) \\ 0, & \text{with probability } 1 - \mathbf{P}(B) \end{cases}$$

- If the random variables, A and B, are independent, we have $P(A \cap B) = P(A)P(B)$. The indicator random variable I_A and I_B, are independent if $P_{I_A,I_B} = P_{I_A}(x)P_{I_B}(y)$
- We know that the intersection of A and B yields:

$$\begin{aligned}
\mathbf{P}_{I_A,I_B}(1,1) &= \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(1) \\
&= \mathbf{P}(A)\mathbf{P}(B) \\
&= \mathbf{P}(A \cap B)
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{I_A,I_B}(1,1) &= \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(1) \\
\mathbf{P}_{I_A,I_B}(0,1) &= \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B) = \mathbf{P}_{I_A}(0)\mathbf{P}_{I_B}(1) \\
\mathbf{P}_{I_A,I_B}(1,0) &= \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c) = \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(0) \\
\mathbf{P}_{I_A,I_B}(0,0) &= \mathbf{P}(A^c \cap B^c) = \mathbf{P}(A^c)\mathbf{P}(B^c) = \mathbf{P}_{I_A}(0)\mathbf{P}_{I_B}(0)
\end{aligned}$$

b) Show that if X = I_A, then E[X] = P(A)

$$E[X] = E[I_A] = 1 * P(A) + 0 * \big(1 - P(A)\big) = P(A)$$