國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 6: counting

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Homework #1

Make sure you turn in your homework

Homework 3/11

Quiz 3/13

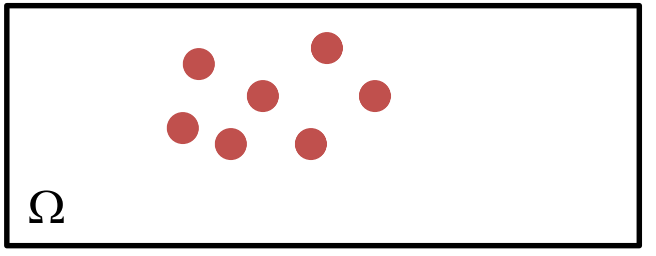
Lecture outline

Readings: Section 1.6

- Principles of counting
- Many examples
  - Permutations
  - Combinations
  - Partitions
- Binomial probabilities

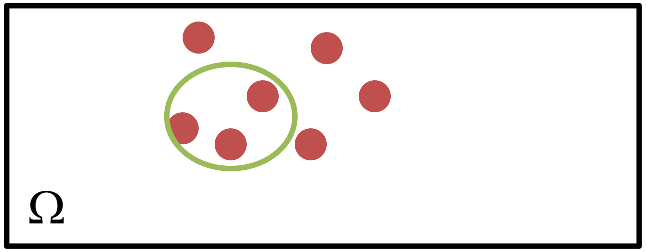
- There is a whole field of mathematics called ‘combinatorics’
- Counting methods apply in situation where we have experiments with a finite number of outcomes, where every outcomes is ‘equally-likely’

- Sample space, with a bunch of discrete points (finite outcomes), with cardinality **N**
- We assume each point is equally likely, which means each outcome has probability: **1/N**



Discrete uniform law

- Given a subset, A, say with cardinality **n**
- To find the probability, essentially add up 1/N probabilities over the cardinality **n**



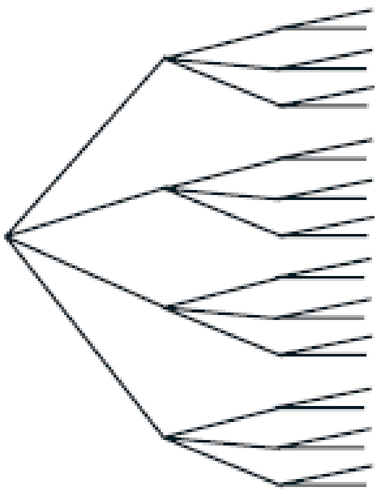
$$P(A) = \frac{\text{number of elements of A}}{\text{total number of elements of } \Omega} = \frac{|A|}{|\Omega|}$$

Just count!

Basic counting principles

Sequential process:

- Number of stages with number of choices at each stage
- Say, r stages, n_i choices at stage *i*



Number of choices: n_1 n_2 n_3 … n_r
Number of leaves = count total number of choices (elements) in this experiment

Number of license plates possible with 3 letters and 4 digits?

Repeatable: 26 * 26 * 26 * 10 * 10 * 10 *10
Non-repeatable: 26 * 25* 24 * 10 * 9 * 8 * 7

- Permutation
  - You have a list of item, number of possible ways order a list
  - [1,2,3,4] -> one way of permutation is [1,4,2,3]
  - So the total number of permutation, given n choice at a time?
  - n (n-1) (n-2) …
  - You've n-way to put it at the first stage, once put it, you have (n-1) way of putting it at the second stage, and continue..
  - n!
  - Count: the number of possible ways you can order *n* elements

- A different question,
  - n elements, {1,2,…n}
  - Count: how many subsets can be created from this set?

- Creating a subset? What does it mean
  - It means deciding whether to include an element in a set (bucket) or not
  - Looking at these in terms of stages?
    - For each stage, you have a binary decision
    - For each element, you have 2 choices (put or not put), n elements
  - The total possible ways: 2^n
  - That's total number of subsets
    - Take example of n=1
    - 2 sets, which 2?

Quick example of rolling a die

- Probability that six rolls of a six-sided die all give different numbers?
  - Let's make some assumptions:
  - Each roll is independent of one another
  - The die is a fair die
  - Then a particular outcome (3,3,1,6,5) has probability $(\frac{1}{6})^6$
  - Any outcome will be the same!

  - Now let's start to work out this problem…

- Event A: all rolls give different number
- So the answer if straightforward: $\frac{|A|}{|\Omega|}$
  - Now let's try to find it:
  - $|\Omega|=6^6$, each outcome has probability $1/6^6$
  - To get different number of 6 rolls, all 6 sides have to show up!
  - Each number can happen at any point in time in this sequence of rolling,
  - So this about it as a list of [1,2,3,4,5,6], and compute the permutation = 6! (|A|)
  - Final answer: $6! / 6^6$

Combinations

We know how many total subsets can be constructed from a set with n elements:

- Let's say we are interested in subsets that have exactly *k* elements
- $\binom{n}{k}$: n choose k = total number of k elements subsets of a given n element set
- How to do: let's think about the problem this way first: how many ways that I can pick k people out and put them in a particular order
  - 1: choose k people, one after another (sequential process)
    - n(n-1)(n-2)…(n-k+1) = $\frac{n!}{(n-k)!}$ choices
  - 2. how many ways we can order this k-choices? (k element permutation)
    - k!

Summarize a little bit:

- Choosing k elements :
- Choose the k items one at a time:

$$n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!} \text{ choices}$$

- Choose k items, then order them (k! possible order)

- Hence:

$$\binom{n}{k} k! = \frac{n!}{(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- These are called ‘binomial coefficients’
- Sanity check
  - Take n=k, should be 1 – only 1 set
  - Take n=0, should also be 1 – the empty set!

Quick example

$$\sum_{k=0}^{n} \binom{n}{k} = ?$$

- Let's think about it
- This is counting the number of k-subsets for different numbers of k
- It's counting the total number of subsets!
  - Answer = 2^n
  - Try to 'interpret' the answer in meanings! Otherwise, the algebra can get messy

Binomial probabilities

- n independent coin tosses
  - P(H) = p
  - So what's the P(HTTHHH)?
- P(sequence) = $p^{\#heads}(1\text{-}p)^{\#tails}$

So, what is the probability that my experiment results in exactly k heads, but in some arbitrary order?

$$P(k \text{ heads}) = \sum_{k-head\ sequence} P(seq)$$

$$= (\#\ of\ k - \text{head seqs}) p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

These are called binomial probabilities

More coin tossing problem now that we know about binomial probabilities

- Event B: 3 out of 10 tosses were "heads"?
  - Given that B has occurred, what is the (conditional) probability that first 2 are heads
  - In the original sample space, are all event equally-likely not necessary!
    - Inside sample space of B, things are equally likely
    - All outcomes: $p^3(1-p)^7$
  - To do this problem: because all outcomes are equally-likely in this new little universe
    - Count $|A|$and $|B|$
  - Number of possible outcomes in B?
    - How many ways we can get 3 heads out of 10 tosses?
    - $\binom{n}{k}$, where n = 10, k = 3

- How many ways A can happen within the universe of B?
  - B has 3 heads in a sequence, A has head-head as starting
  - So the only uncertainty of A lies in where the THIRD head is going to be located?
    - There are ‘8 other places’ that this can happen => so 8 different ways!

- So what’s the answer to the question:

Given that B has occurred, what is the (conditional) probability that first 2 are heads

$$\frac{8}{\binom{10}{3}}$$

Partitions

- A little harder problem, how many ways to partition:
  - 'n choose k' , thinking about it that there are n-elements, and picking k out, which means there are n-k left
  - Picking a subset means partitioning the original space into two pieces, say instead of 2 partitions,
  - You are told a certain number, n_1, n_2, n_3, n_4, where $n_1+n_2+n_3+n_4 = n$, (4 ways – partitions to complete the sample space) how many ways can we do this partitions?

  - Let's imagine a concrete case: card-dealing
  - 52-card deck, dealt to 4 players

- Assume the dealing is done fairly with a well-shuffled deck of card
  - Every possible partition of the 52 cards into 4 subsets should be equally-likely
  - Back to counting again

Find the P(each gets an ace)

- Outcomes (samples space elements) of this experiment: a partition of 52 cards
  - Total number of outcomes

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{39!\,13!} * \frac{39!}{26!\,13!} * \frac{26!}{13!\,13!} * 1$$

$$= \frac{52!}{13!\,13!\,13!\,13!}$$

- Event of interest: everyone gets an ace
  - How many ways can that happen?
  - By taking 4 aces to each player = 4*3*2*1
  - Now the remaining 48 cards, what are possible ways to dealing them to each of the 4 players (12 cards each)

$$\frac{48!}{12!\,12!\,12!\,12!}$$

So what's the answer P(each player get an ace after being dealt with 52 cards)

$$\frac{4 * 3 * 2 * \frac{48!}{12!\,12!\,12!\,12!}}{\frac{52!}{13!\,13!\,13!\,13!}}$$

P1 birthday problem

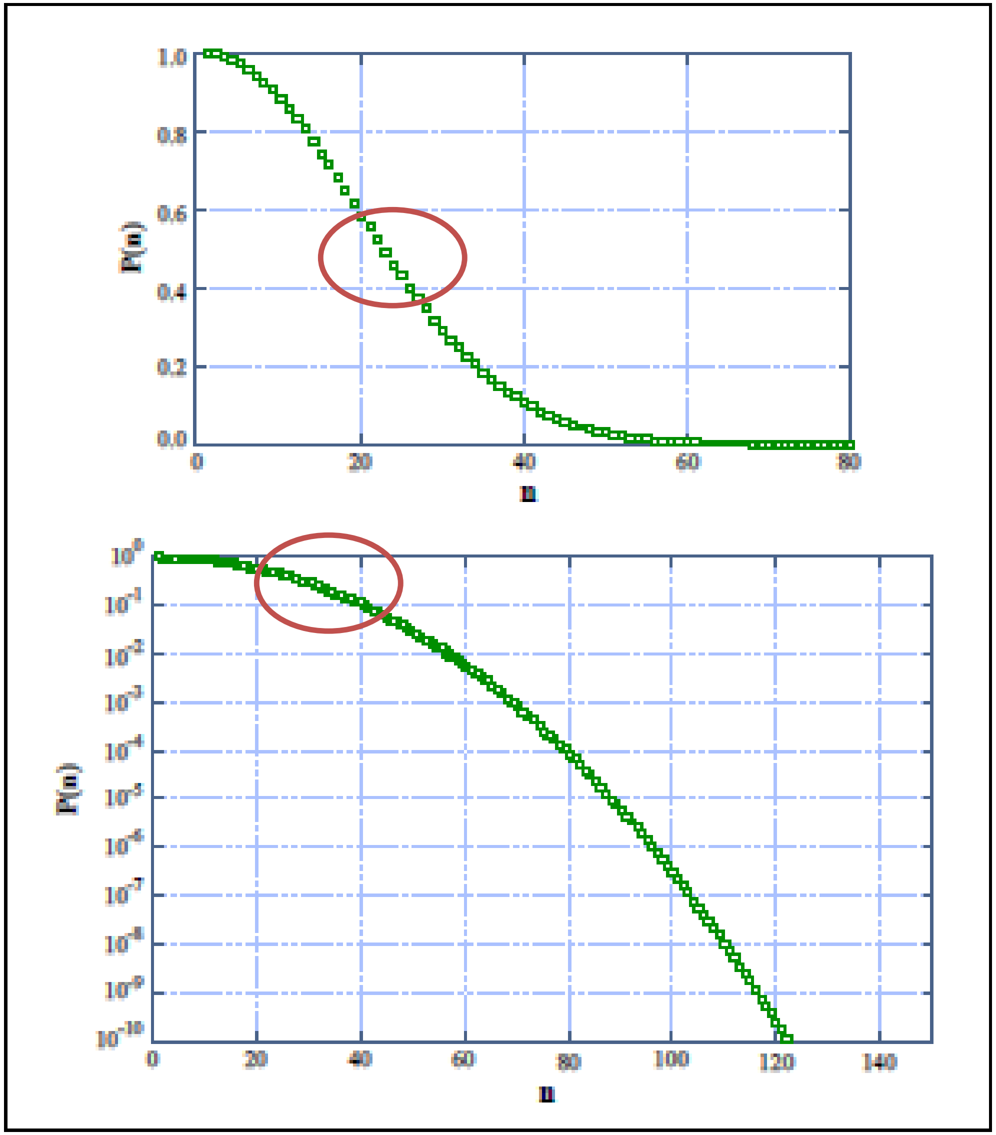
Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independently of everyone else, and ignore the additional complication presented by leap years (i.e., nobody is born on February 29). What is the probability that each person has a distinct birthday?

The sample space consists of all possible choices for the birthday of each person. Since there are n persons, each has 365 choices for their birthday:

The sample space has 365^n elements.

P(no two birthdays coincide)?

- Number of choices for which no two persons have the same birthday: 365 * 364 * 363 *… (365-n+1)
- The probability equals: $\frac{365 * 364 * \cdots * (365-n+1)}{365^n}$



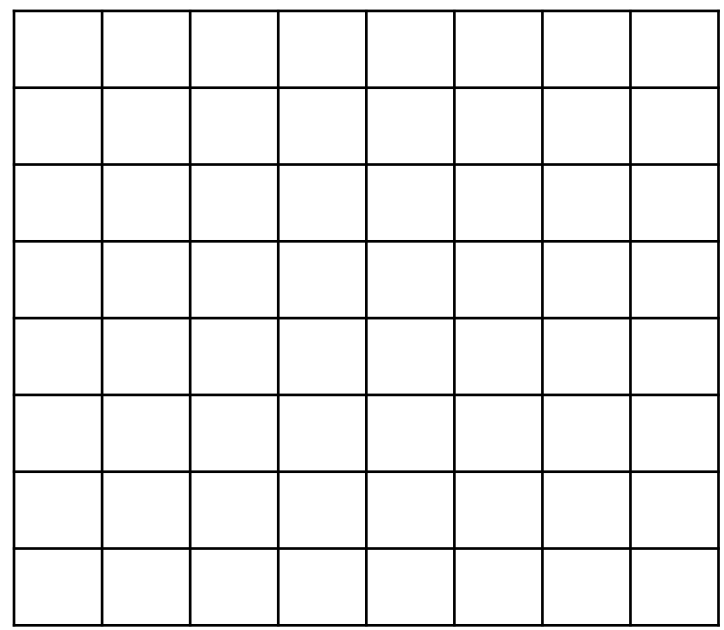
- It is interesting to note that for n as small as 23, the probability that there are two person with the same birthday is larger than 1/2

p2

Imagine that 8 rooks are randomly placed on a chessboard. Find the probability that all the rooks will be safe from one another, i.e., there is no row or column with more than one rook.

- First: a chessboard is 8x8 (64 possible positions!)
- So what we need is to count the number of ways that we can put these 8 rooks on ‘favorable positions’
  - First rook: there are not constraints, so 64 options; that removes 1 column and 1 row
  - Only 7x7 left, so 49 choices, then 6x6, 36 choices, … similar for third rook and so on
  - Total possible ways to put 8 rooks on the board without constraint? 64*63*…*57 = 64!/56!

$$\frac{64 * 49 * 36 * 25 * 16 * 9 * 4}{\frac{64!}{56!}}$$



p3

An urn contains n balls, out of which m are red. We select k of the balls at random, without replacement (i.e., selected balls are not put back into the urn before the next selection). What is the probability that i of the selected balls are red?

The sample space?

- $\binom{n}{k}$ possible ways to choose k items out of the available balls

For the events of interest to occur, we have to select *i* out of the *m* red balls

- $\binom{m}{i}$ ways to do it
- To select *k-i* out of *n-m* balls that are not red, this can be done in $\binom{n-m}{k-i}$ ways (partitions..)

The desired probability is

$$\frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}}$$

This is also called “hypergeometric probability”

Quick review

Now you have learned it all: Steps to make probabilistic reasoning

- Setup sample space
- Statement about probability law
- Identify events, and calculate probability for those outcomes of interest

Sample Space Ω

- "List" (set) of possible outcomes

- Requirements:
  - Mutually exclusive
  - Collectively exhaustive

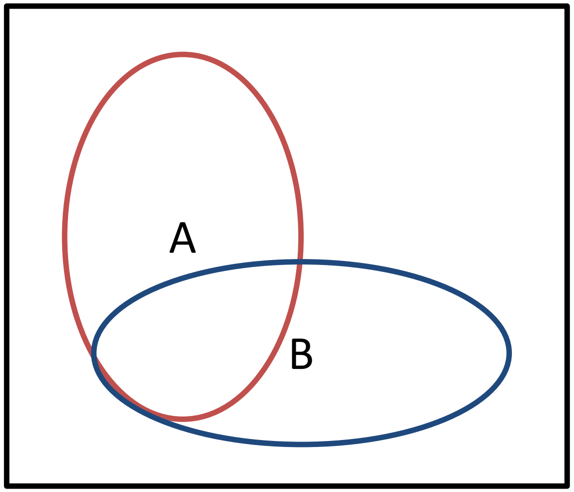
- Art: to be at the "right" granularity
  - Pick (decide) appropriate sample space for your problem
  - Let's think about '*coin-flipping*' exp. again
  - Real world, real life

Countable additivity axiom

If A1, A2, …are disjoint then:

$$P(A_1 \cup A_2 \cup A_3 \cup, \ldots) = P(A_1) + P(A_2) + \cdots$$

Definition of conditional probability



- P(A|B) = probability of A, given that B occurred
  - B is our new universe
- Definition: assuming P(B) $\neq$ 0

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  - P(A|B) undefined if P(B) = 0
  - We can compute the previous example using this equation

Generalize these calculations I
multiplication rule

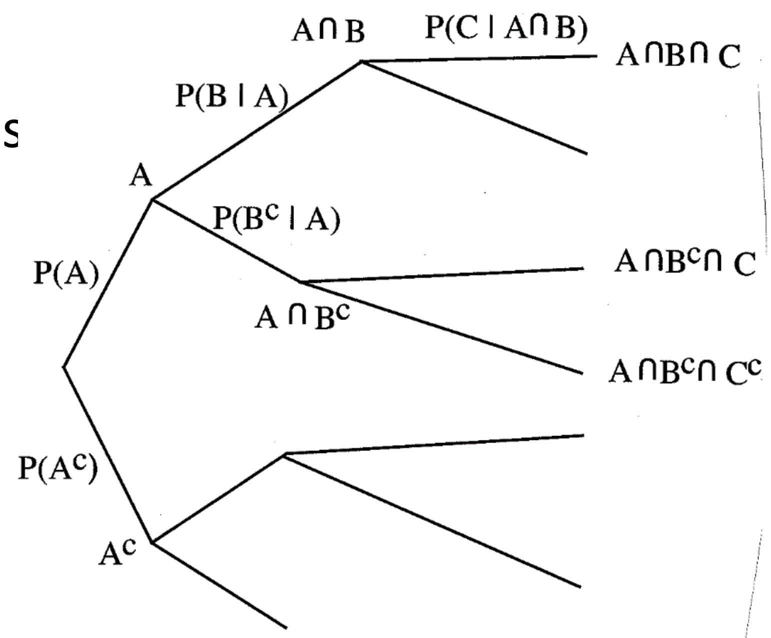
Find a composite event's probability

- Multiply probabilities and conditional probabilities
- $P(A \cap B) = P(B|A) * P(A)$

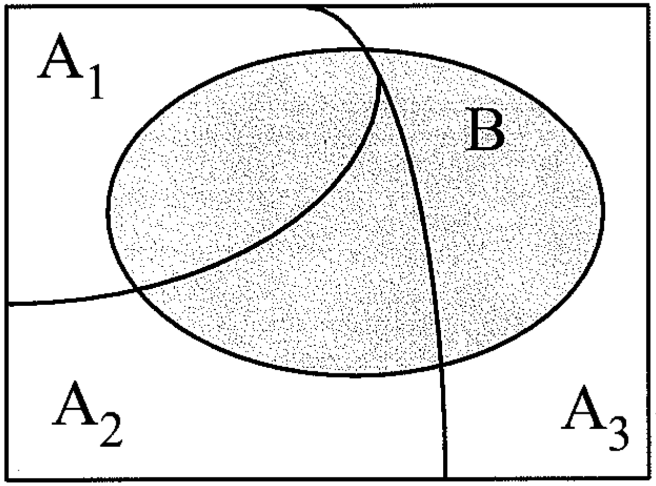
Generalize this to multiple events (A, B, C):

$P(A \cap B \cap C) = P(A)\, P(B|A)\, P(C|A \cap B)$

Essentially, multiplying through the leaves



Total probability theorem



- Divide and conquer
  - Partitions split entire sample space
  - Compute $P(B|A_i)$

To compute P(B) above:

P(B) = P(A1) P(B|A1)
+ P(A2)P(B|A2)
+ P(A3)P(B|A3)

Bayes Rule

- Inference problem
  - Remember, cause-effect

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{assuming P(B)>0}$$

- Multiplication rule:

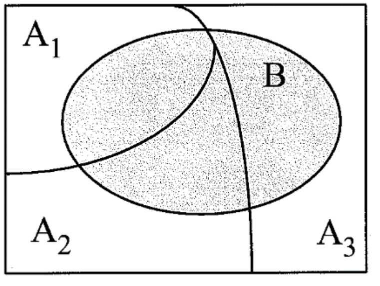
$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

- Total probability theorem:

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

- Bayes rule:

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$$



Independence of two events

Definition: $P(B|A) = P(B)$

- Occurrence of A provide no information about B's occurrences
- Conditional probability = unconditional probability
- Recall that $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$

Alternative definition: $P(A \cap B) = P(A)P(B)$

- Symmetric with respect to A and B
- Applies even if P(A) = 0
- Implies P(A|B) = P(A)

- Independence can be checked mathematically (and should be!)
  - It can also be checked (argued) with reasoning