國立清華大學
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EE 306001
Probability

Lecture 5: Independence

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Lecture Outline

Reading: Section 1.5

- Independence of two events
- Independence of a collection of events

Let's review a simple model again

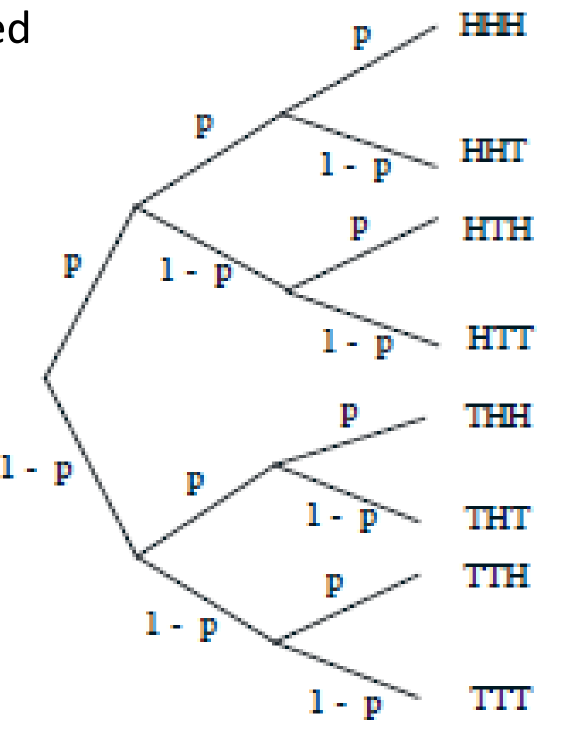
Using conditional probability to construct a model

One experiment = three consecutive tosses of a biased coin

P(H) = p, P(T) = 1- p

The tree diagram here is a conditional probability

In terms of this experiment, each toss of obtaining a {H} or {T} does not change with respect to what happened at the pervious toss



We are going to talk about this in more details today = Concept of independence

Independence of two events

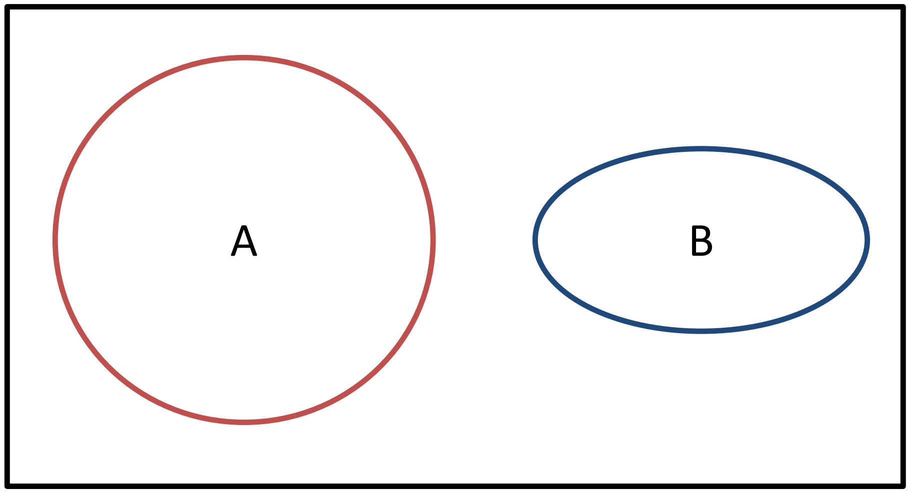
Definition: $P(B|A) = P(B)$

- Occurrence of A provide no information about B's occurrences
- Conditional probability = unconditional probability
- Recall that $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$

Alternative definition: $P(A \cap B) = P(A)P(B)$

- Symmetric with respect to A and B
- Applies even if P(A) = 0
- Implies P(A|B) = P(A)

- Independence can be checked mathematically (and should be!)
  - It can also be checked (argued) with reasoning



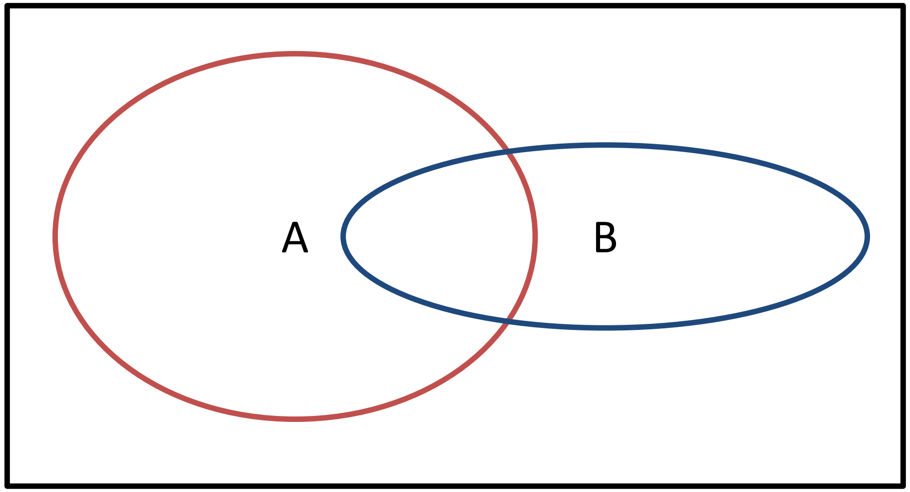
Are these two events independent?

They are disjoint and separated, but not independent!

If A occurs, B definitely does not occur

The occurrence of one event does in fact change your belief

Don’t confuse disjointness with independence



- Venn diagram is not enough for checking independence
- Use mathematical definition to check
  - $P(A \cap B) = P(A)P(B)$

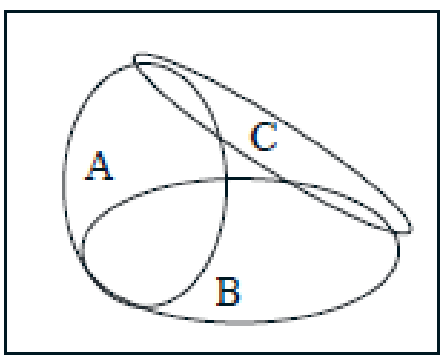
Conditioning affects independence

First of all,

Conditional probability is just like ordinary probability,

Definition of independence is similar too

$$P(A \cap B|C) = P(A|C)P(B|C)$$



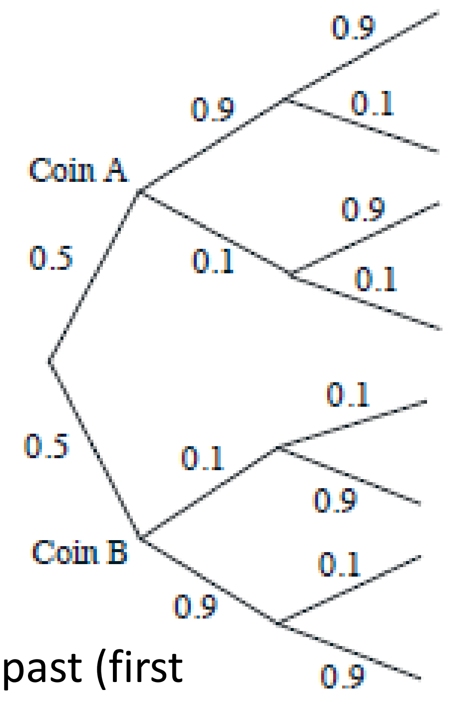
Assume event A and event B are independent

If we are told that C has occurred
Are A and B still independent?

Nope! Two pieces become disjoint!
Conditions may change from independent to dependent

Let’s see another examples

- Two unfair coins, A and B:
  - P(H|coin A) = 0.9,
  - P(H|coin B) = 0.1
- Experiment of flipping each coin: each toss is independent
- Extend the experiment: choose a coin at random, and start flipping



If we don’t know which coin is it and what happened in the past (first 10 tosses), what is the probability of 11th tosses to be head

P (toss 11 = H) = 0.5*0.9 + 0.5 * 0.1 = ½

However, What if we observe 10 heads in row!

P(toss 11 = H | 10 heads in a row)

Knowing this information does it change your belief now?

It should!

Obtaining 10 heads in a row, which is the coin that you choose?
Coin A

So you are quite certain know it’s more coin A, then that modifies your belief!

The ‘inference’ on which it could have been changes you belief (individual toss is then no longer independent) – Look at the problem description carefully

(the event of coin toss is independent?)

Independence of a collection of events

- Intuitive definition:

Information on some of the events tell us nothing about the probabilities related to the remaining events:

e.g., $P(A1 \cap (A2 \cup A3)|A5 \cap A6) = P\big(A1 \cap (A2 \cup A3)\big)$

- Cleaner/more precise definition:
  - Event A_1, A_2, …, A_n are independent

    $P\big(A_i \cap A_j \cap \cdots \cap A_q\big) = P(A_i)P\big(A_j\big) \ldots P(A_q)$

    for any distinct indices, i, j, …q

    indices go from 1…n

Pairwise independence vs. independence

- Two independent fair coin toss
  - A: first toss is H
  - B: second toss is H
  - P(A) = P(B) = ½
  - Are these two events independent? Yes!
- C: first and second toss give the same result
  - P(C) = ½
  - $P(C \cap A) = P(A) * P(C|A)$
    - P(C)= ½, P(C|A)=1/2
    - P(C)*P(A)= ¼
  - $P(C \cap A \cap B)$ = ¼
    - P(C)*P(A)*P(B)=1/8
  - $P(C|A \cap B) = 1$

HH	HT
TH	TT

A and B and C are all pair-wise independent
Telling you each does not affect your belief on any one!
However, telling you that A and B both occurs changes the story!
Independent needs to be check with more thoughts!

P1 The kings sibling

- The king comes from a family of two children. What is the probability that his sibling is female?
  - The king is a boy! – one of those ancient world's rule

- A fast intuitive thought
  - P(boy) = P(girl) = ½
  - Having a boy and a girl is equally likely, and two child-birth are two independent event!

- Now, let's formally do it:
  - Experiment has 4 outcomes {boyboy, boygirl, girlboy, girlgirl}
  - Now we are told there is a king
    - It means that the outcomes {girlgirl} didn't happen
    - So P(king's sibling is a girl) = 2/3! – not 1/2

- Okay, more complicated
  - this answer is assuming that the 'reproductive strategy' is that King's parent is determined to have two kids

- If the strategy is 'going to have children until we get a boy', if there are two kids, what's the probability that kings' sibling is a girl? 1!

Imagine a drunk tightrope walker, in the middle of a really long tightrope, who manages to keep his balance, but takes a step forward with probability p and take a step back with probability, 1-p

What is the probability that after two steps the tightrope walker will be at the same step on the rope

- In order to wind up in the same place after two steps, the tightrope walker can either step forwards, then backwards, or vice versa,

Therefore, the required probability is (every step is an independent step):

$$2p(1-p)$$

What is the probability that after three steps, the tightrope walker will be one step forward from where he began?

- The probability that after three steps he will be one step ahead of his starting point is the probability that out of 3 steps in total, 2 of them are forwards, and one is backwards.

This equals:

$$p * p * (1 - p) * 3 = 3p^2(1 - p)$$

(F F B), (F B F), (B, F, F)

Given that after three steps he has managed to move ahead one step, what is the probability that the first step he took was a step forward?

- Given that out of his three steps only one is backwards, the sample space for the experiment now becomes:

- P(first step is forward|after three step he moved one step ahead)

after conditioning: the sample space become the following

{(F, F,B); (F,B, F); (B, F, F)}

Answer: 2/3

P3. Let A and B be events such that $A \subset B$. Can A and B be independent?

If $A \subset B$, then $P(B \cap A) = P(A)$

But we know that in order for A and B to be independent, $P(B \cap A) = P(A)P(B)$.

Therefore, A and B are independent if and only if $P(B) = 1$or $P(A) = 0$.

This could happen, for example, if B is the universe or if A is empty.

P4

The local widget factory is having a blowout widget sale. Everything must go, old and new. The factory has 500 old widgets, and 1500 new widgets in stock. The problem is that 15% of the old widgets are defective, and 5% of the new ones are defective as well. You can assume that widgets are selected at random when an order comes in. You are the first customer since the sale was announced.

a) You flip a fair coin once to decide whether to buy old or new widgets. You order two widgets of the same type, chosen based on the outcome of the coin toss. What is the probability that they will both be defective?

Suppose we choose old widget, before we do anything, there are 500 * 0.15 = 75 defective old widgets. The probability that we choose two defective widgets is

P(two defective|old) = P(first is defective|old) P(second is defective| first is defective, old)
75/500 * 74/499 = 0.02224

Now let's consider the new widgets, we can do the similar calculation (1500*0.05=75)
75/1500 * 74/1499 = 0.002568

By the total probability law,

P(two defective) = P(old) P(two defective|old) + P(new)P(two defective|new)

0.5 * 0.02224 + 0.5 * 0.002568 = 0.01240

- Given that both widgets turn out to be defective, what is the probability that they were old widgets?

Using Bayes Rule:

$$P(\text{old}|\text{two defective}) = \frac{\text{P(old)P(two defective|old)}}{P(\text{old})P(\text{two defective|old}) + P(\text{new})P(\text{two defective|new})}$$

$$= \frac{0.5 * 0.02224}{0.5 * 0.02224 + 0.5 * 0.002568} = 0.8965$$

P4. Monte hall problem: quite a famous game

You are told that a prize is equally-likely to be found behind any one of the three closed doors in front of you. You point to one of the doors. A friend opens for you one of the remaining two doors, after making sure that the prize is not behind it.

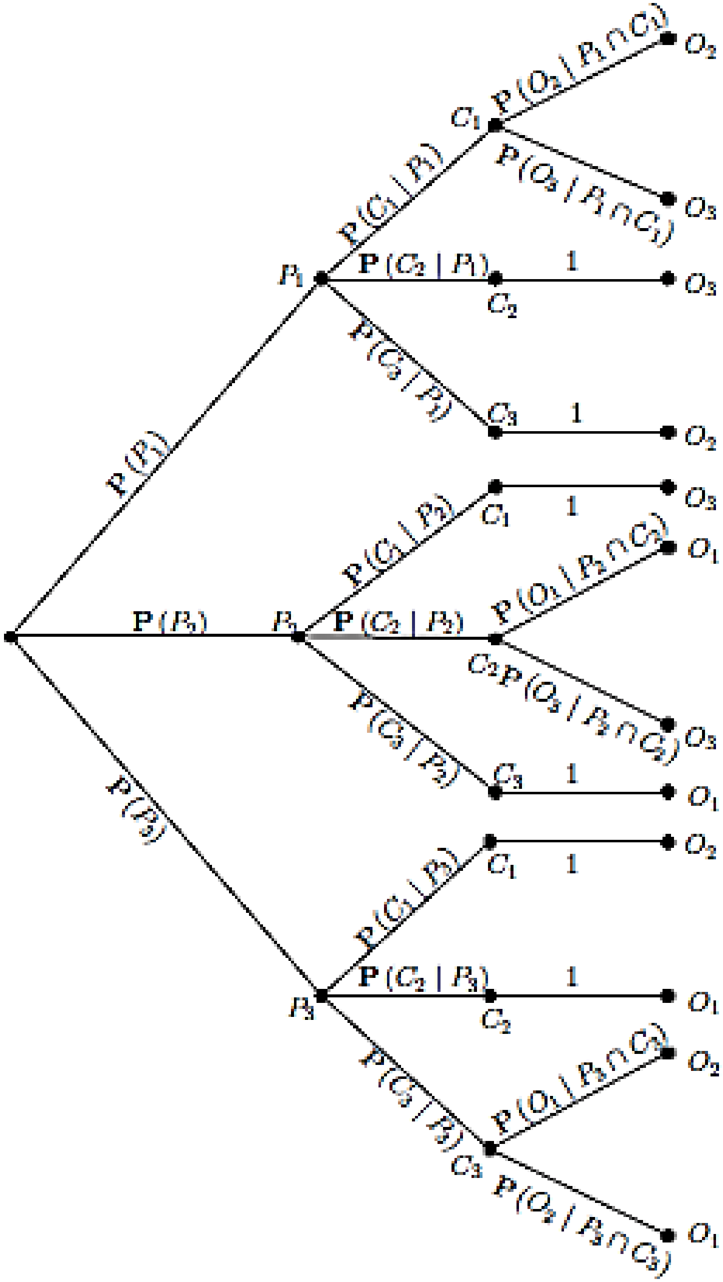
At this point, you can stick to your initial choice, or switch to the other unopened door, how should we choose?

1) Stick to the initial choice
2) Switch to the other unopened door

Let us write down some events of interest in this experiment:

- P_i denote the event where the prize is behind door *i*
- C_i denote the event where you initially choose door *i*
- O_i denote the event where your friend open the door *i*

Once having this: we can write out the entire probability tree



1) Stick to the initial choice

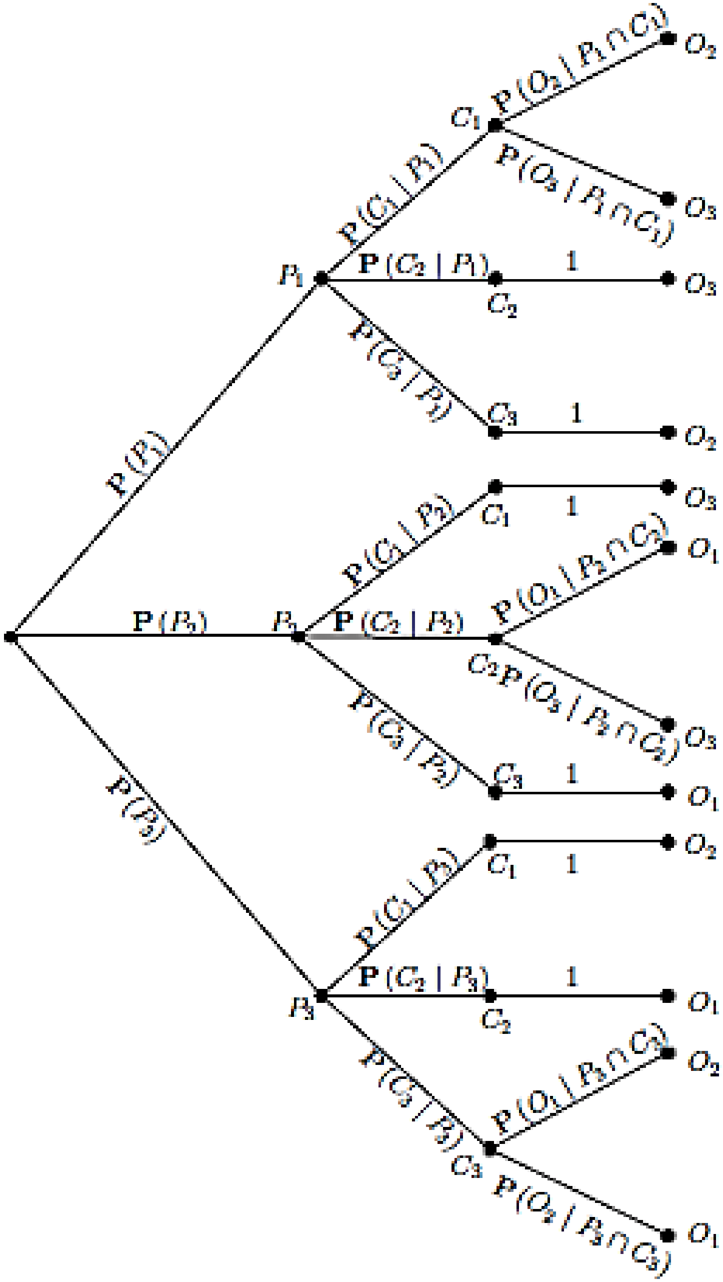
The probability of winning when not switching is the probability that the prize is behind the original door that you choose

$P(P1 \cap C1) + P(P2 \cap C2) + P(P3 \cap C3)$
$= P(P1)\ P(C1|P1) + P(P2)\ P(C2|P2) + P(P3)\ P(C3|P2)$
$=P(P1)P(C1)+P(P2)P(C2)+P(P3)P(C3)$
$= 1/3$

P(C1|P1)=P(C1)

Whether the price is behind door 1 has nothing to do with what you pick at the beginning

Each also is equally-likely (1/3)



$P(P1 \cap C2 \cap O3)$
$= P(P1 \cap C2)$
O3 is deterministic

2) Switch to the other unopened door

The probability of winning when switching is the probability that the prize is behind the unopened door that you choose

P(Winning when switching)

$= P(P1 \cap C2 \cap O3) + P(P1 \cap C3 \cap O2) + P(P2 \cap C1 \cap O3) + P(P2 \cap C3 \cap O1) + P(P3 \cap C1 \cap O2) + P(P3 \cap C2 \cap O1)$

$= P(P1 \cap C2) + P(P1 \cap C3) + P(P2 \cap C1) + P(P2 \cap C3) + P(P3 \cap C1) + P(P3 \cap C2)$

$= P(P1)\,P(C2) + P(P1)\,P(C3) + P(P2)\,P(C1) + P(P2)P(C3) + P(P3)P(C1) + P(P3)\,P(C2)$

$= 2/3$

SWITCHING IS BETTER!

Next lecture

About counting,

Why counting: applies to most finite outcome experiment, where each outcome is ‘equally-likely’

So the question often boils down to counting (combinatorics)

- This is a hard field!
- Often, in real world practice, counting is not the ‘major’ part of probability applications…