國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 4: problem solving

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Review

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{assuming P(B)>0}$$

- Multiplication rule:

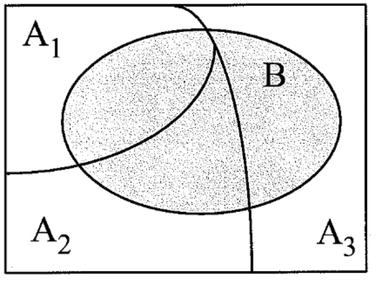
$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

- Total probability theorem:

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

- Bayes rule:

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$$



P1.

A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that first toss is a head than if we know at least one of the tosses is a head

- Is she right?
- Does it make a difference if the coin is fair or unfair
- How can we generalize Alice’s reasoning

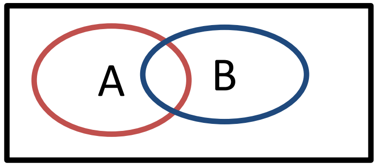
- Let A be the event that the <u>first</u> toss is a head
- Let B be the event that the <u>second</u> toss is a head

Is the claim correct?

What we need is to compare conditional probabilities

$$P(A \cap B | A) \text{ vs. } P(A \cap B | A \cup B)$$

$$P(A \cap B | A) = \frac{P((A \cap B) \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$



$$P(A \cap B | A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}$$

Since $P(A \cup B) \geq P(A)$

$P(A \cap B | A)$ is at least as large $P(A \cap B | A \cup B)$

Yes! The claim is correct

Does it matter if the coin is fair or not? No~

But, if we know that the coin is fair, we can do an actual computation to get exact number!

Four outcomes: {HH}, {HT}, {TH}, {TT} are equally likely

$$\frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\frac{P(A \cap B)}{P(A \cup B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Generalize Alice's claim:

Say if we have three events, A, B, C, such that:

$B \subset C$

$A \cap B = A \cap C$ (for example, $A \subset B \subset C$)

Then,

Event A is at least as likely if we know that B has occurred than if we know that C has occurred

In the case of this experiment:

$C = A \cup B$

Conditioning on a smaller set : changes the sample space to a smaller one
Provide 'more' information

P2.

We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely

a) Find the probability that doubles are rolled

Doubles:

(1,1), (2,2), (3,3), (4,4), (5,5,), (6,6)

6/36 = 1/6

b) Given that the roll results in a sum of 4 or less, find the conditional prob. that doubles are rolled

What are the event that results in a sum of 4 or less:

We can easily list them all out:

{1,1}, {1,2},{1,3},{2,1},{2,2},{3,1}

How many of them are doubles?

Just 2 of them

P(doubles are rolled | roll results in a sum of 4 or less) = 2/6

c) Find the probability that at least one die roll is a 6

Let’s do more counting,

Out of 2 roll of 6 sided dice, how many of outcomes will result in having at least 1 die roll is 6?

(6,6), (6,i), (i,6), where i = 1,2,3,4,5

There are 11 possibilities

P(at least one die roll is 6) = 11/36

d) Given that two dice land on different numbers, find the probability that at least one die roll is a 6

This is a conditional probability calculation!

- Total number of possibilities is 36
  - Landing on different number (36 – 6 (doubles)) = 30 possible outcomes

- Out of these 30 possibilities, how many of them has at least one die roll 6?
  - (6,i), (i,6) , i=1,2,3,4,5 (10 total outcomes)
  - P(at least one roll is 6 | two dice land on different numbers) = 1/3

P3.

You enter a chess tournament where your probability of winning a game is 0.3 against half of the player (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3).

Now you play against a randomly chosen opponent

a) What is the probability of winning

Let A_i be the event of playing against opponent of type *i*

$P(A_1) = 0.5$, $P(A_2) = 0.25$, $P(A_3) = 0.25$

Now, let B be the event of winning, we have

$P(B|A_1) = 0.3$, $P(B|A_2)=0.4$, $P(B|A_3) = 0.5$

Now, remember the total probability theorem

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$
$$= 0.375$$