國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 3: conditional probability

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Lecture Outline

Reading: Section 1.3, 1.4

- Review
- Conditional probability
- Three **important** tools:
  - Multiplication rule
  - Total probability theorem
  - Bayes rule

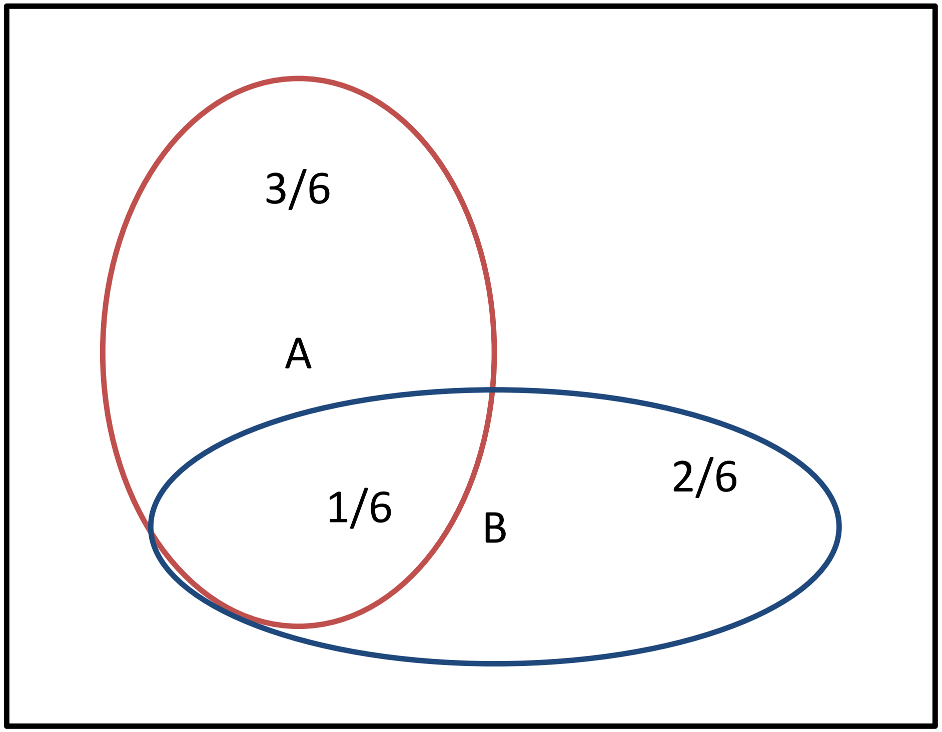
Main agenda for today

At first, you know something about the world of which you write down your belief (probability) for different outcomes

Then, something happens, somebody tells you a little more about the world, this information, should change your 'belief' on your original probabilistic model

Partial information about the outcome of the experiment should revise our beliefs

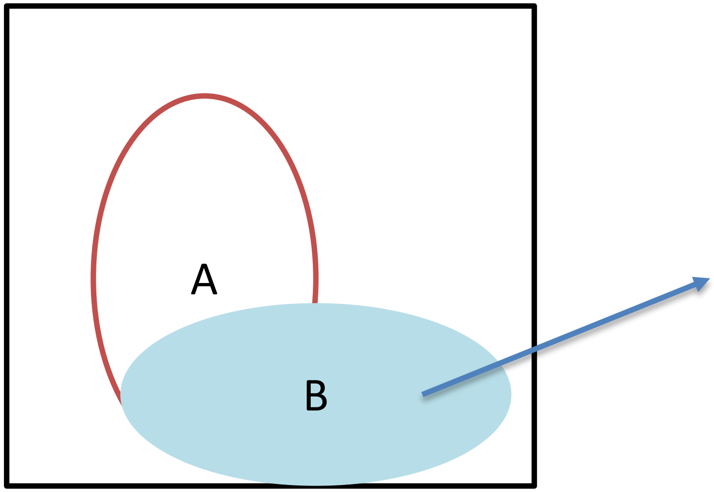
Conditional probabilities are probabilities that apply after the revision of our beliefs, when we are given some information



Somebody just told you, that event B occurs (didn't mention the full outcomes, but just that event B occurs)

Does it change your probabilistic model?

It should!



Let's think intuitively,

Event B originally has probability (2+1)/6 = ½

What is probability of P(B) given that we know outcome lies in event B?

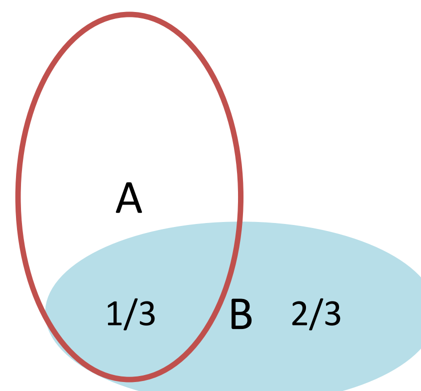
= 1!

- It becomes the new sample space for this experiment

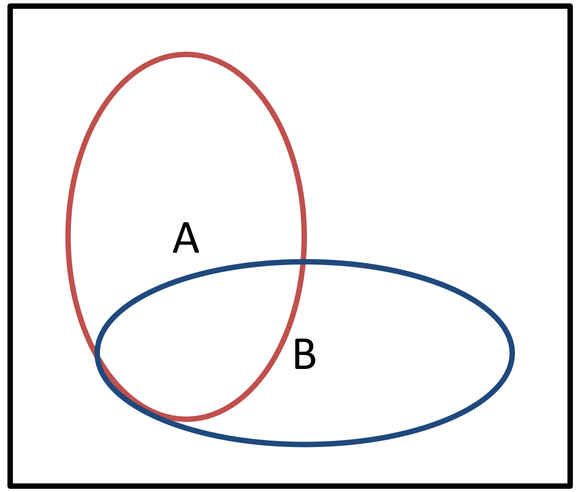
So what is the probability of event A now that we know outcome lie in event B?

- We know originally, within event B, the portion that event A would occur is two times <u>**less**</u> than (event B **AND not** event A)

= 1/3 ! We have revised probability of event A



Definition of conditional probability



- P(A|B) = probability of A, given that B occurred
  - B is our new universe
- Definition: assuming P(B) $\neq$ 0

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  - P(A|B) undefined if P(B) = 0
  - We can compute the previous example using this equation

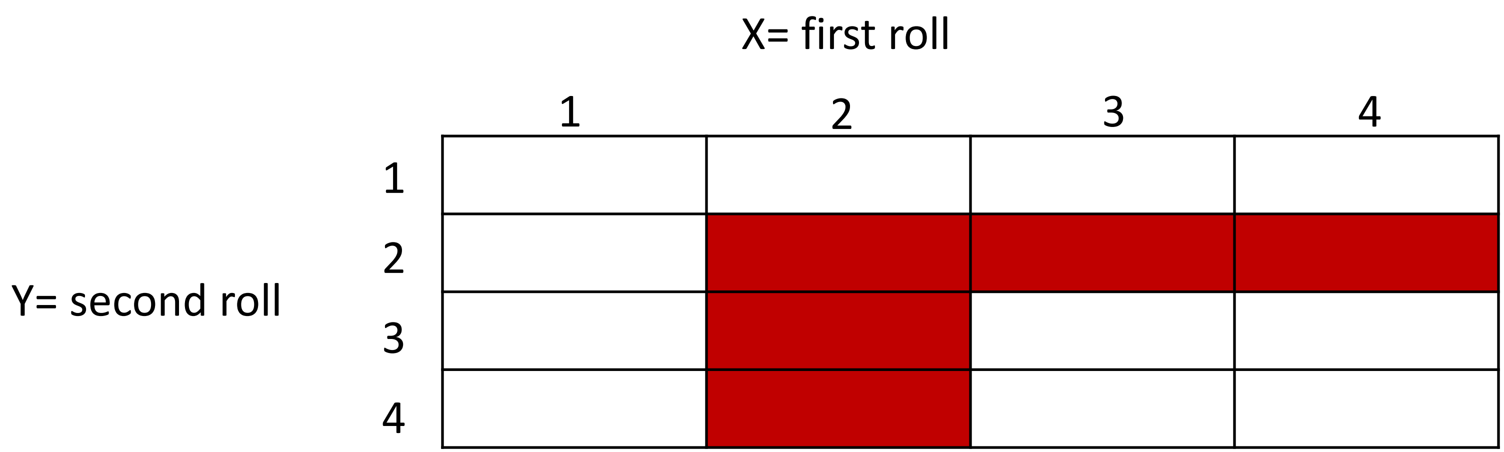
So how should we interpret this equation:

Basically, only look at experiments at which B happens to occur, and then look at what fraction of those experiments where B ALREADY occurred, event A also occurs

There is symmetrical version of this equality for A and B

Conditional probability is just like ordinary probability

- Satisfy all probability law described previously
- You can imagine we just change the universe (sample space) since we are ‘conditioning’ (knowing) that some event has already occurred



Let B be the event: min(X, Y) = 2

Let M = max(X, Y)

- P(M = 1 | B) = 0

Now, lets think about a larger model

- P(M = 2 | B) =

Use definition: $P(A|B) = P(A \cap B) / P(B)$

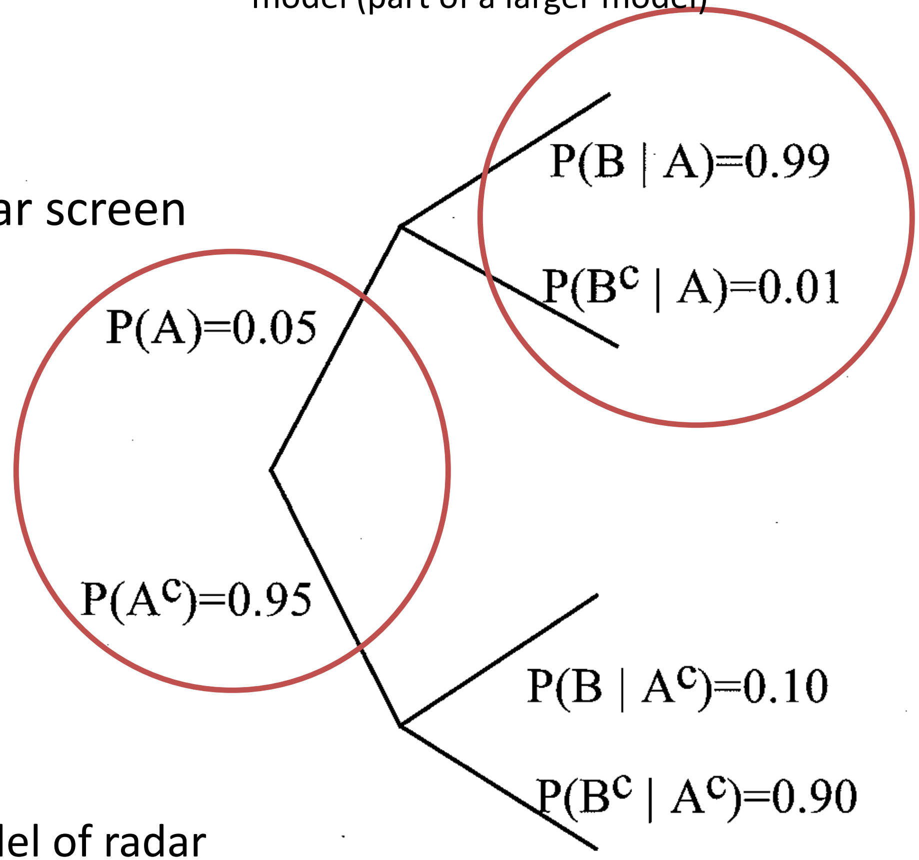
  - P(M = 2 **AND** min(X, Y)= 2) = 1/16
  - P(B) = 5/16
  - P(M = 2 | B) = 1/5

Probabilistic model defined directly from conditional probability

each piece is an individual probabilistic model (part of a larger model)

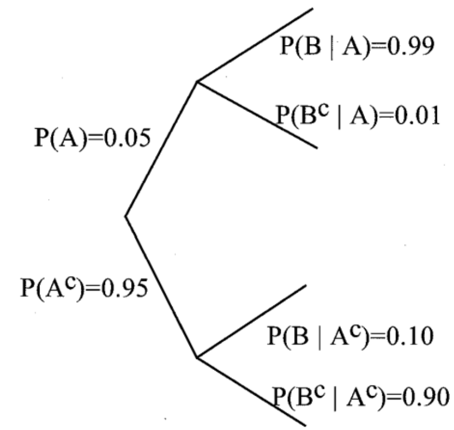
Imagining an airplane on the sky

Event A: Airplane is flying above
Event B: Something registers on radar screen



Now we have specified the complete model of radar detection of an airplane on the sky using conditional probability, we can start answering some question!

- $P(A \cap B)$ =
  - A plane is up there and radar picks it up
  - $P(B|A) = P(A \cap B)/P(A) \Rightarrow P(A \cap B) = 0.99 * 0.05 = 0.0495$
  - We calculate this probability by multiplying through the leaves



- $P(B)$ =
  - what is the probability that radar is going to register something?
  - 2 way:
    - There is a plane & registered
    - There is not a plane & registered
    - 0.0495 + (0.95 * 0.10) = 0.1445

- $P(A|B)$ =
  - Given your radar records something, how likely is it that there is an airplane up there?
  - P(A & B) / P(B) = 0.34

Look at this closely

- P(A|B) =
  - Given your radar records something, how likely is it that there is an airplane up there?
  - Answer is roughly 30%!

Not very high ! Why?

Mostly because of the false alarm!

Conditional probability can have weird (non-intuitive) interpretation something!

Need to be extremely careful about this
e.g, in medical testing – be aware of false alarm

Generalize these calculations I
multiplication rule

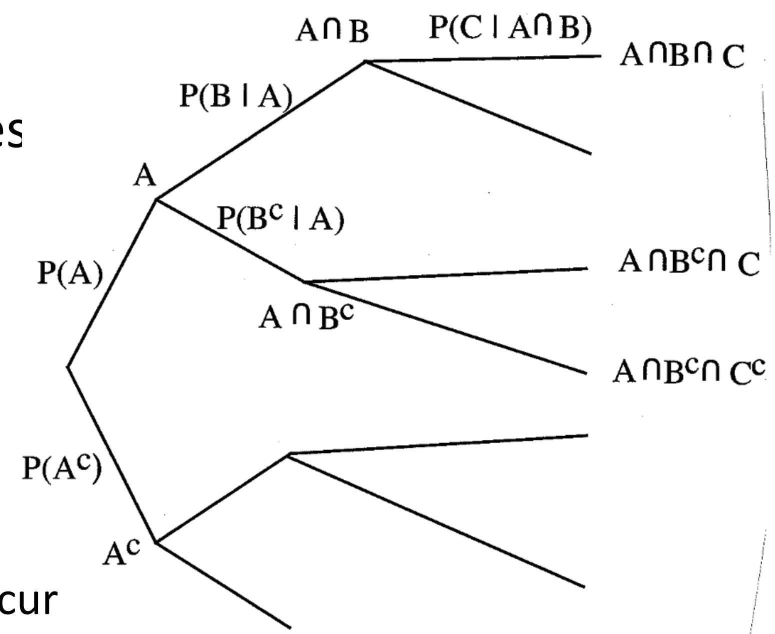
Find a composite event's probability

- Multiply probabilities and conditional probabilities
- $P(A \cap B) = P(B|A) * P(A)$

Generalize this to multiple events (A, B, C):

$P(A \cap B \cap C) = P(A)\, P(B|A)\, P(C|A \cap B)$

Essentially, multiplying through the leaves



Think in terms of frequencies,

How often do three things (A,B,C) occur?

1. How often A occur
2. Out of all times A occur, how often B occur
3. Out of all times A, B occur, how often does C occur

Can we prove this?

- Yeah, straight forward from conditional probability definition

$$P(A \cap B) = P(B|A) * P(A)$$

Here, we have 3 events,

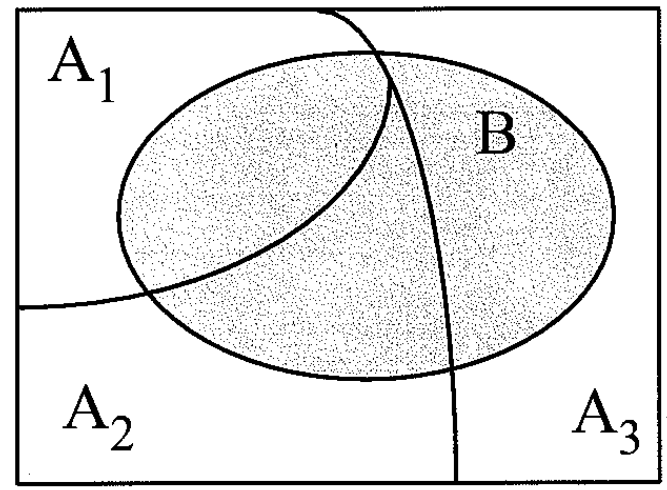
- Simply make $B' = (B \cap C)$
- $P(A \cap B \cap C) = P(A \cap B') = P(A) * P(B' \mid A) = P(A) * P(B \cap C|A)$

$\underline{= P(A) * P(B|A) * P(C|A \cap B)}$

- You can now compute any probabilities on the previous tree diagram using conditional probability

Generalize these calculations II
Total probability theorem

- The previous radar example, we compute P(B)!



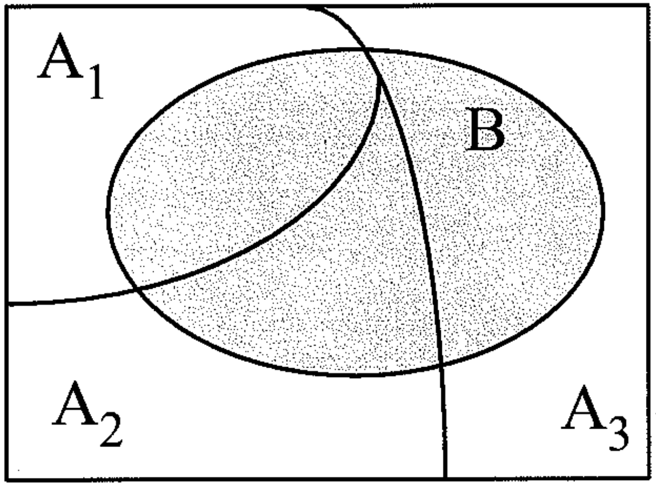
Let's generalize this:

- Find the occurrences of B under different scenarios (A_i) – plane is there or not!

To find the total probability of B

- We add up the probabilities of B occurring under each of the partitions of A

Total probability theorem



- Divide and conquer
  - Partitions split entire sample space
  - Compute $P(B|A_i)$

To compute P(B) above:

$$P(B) = P(A1)\,P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3)$$

Generalize these calculations III
Bayes' rule

- $P(A|B) =$
  - Given your radar records something, how likely is it that there is an airplane up there?
  - An inference problem

Different scenarios = different 'prior' probabilities (initial belief on how likely each scenario is going to occur)

$$P(A_i)$$

We also have model of our measuring device that tells us under that scenario how likely event B (radar registered) is going to happen

$$P(B|A_i)$$

- P(A|B)
  - So what are we trying to infer here?
  - Whether an airplane was present
    - Initially, we believe 5% at any time (now 34%, if you think in this term the radar is not so bad now~)
    - Now we see an additional information/observation (radar recording), how would that change our belief?

How do we do it?

- Apply definition of conditional probability, multiplication rule, total probability theorem!

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

Revert the probabilistic model to perform inference based on 'data' observed!
Cause – effect, we see the effect, infer the cause?
1700's Thomas Bayes