國立清華大學
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EE 306001
Probability

Lecture 2

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Review of probability models

- Sample space
  - Mutually exclusive
  - Collectively exhaustive
  - Right granularity

- Event: subset of sample space
- Allocation of probabilities to event
  - $P(A) \geq 0$
  - $P(\Omega) = 1$
  - If $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
    - If A1, A2, A3, … are disjoint events, $P(A1 \cup A2 \ldots) = P(A1) + P(A2) + \ldots$

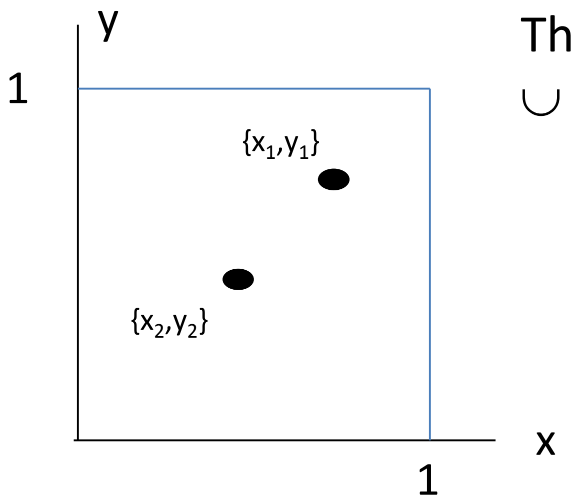
infinite collections of sets

Countable additivity axiom

If A1, A2, …are disjoint then:

$$P(A_1 \cup A_2 \cup A_3 \cup, \ldots) = P(A_1) + P(A_2) + \cdots$$

Sequence of sets (events), occurred in order, and countable, sequence of sets disjoint



The sample space of this unit square:
$\cup$ of one-element set (one element = one point (x,y))

According to axiom 2:

$$P(\Omega) = 1$$

According to additivity axiom

$$P(A_1, A_2,\ldots) = P(A_1) + P(A_2)+\ldots = 0$$

1 = 0? What is the problem here?

<u>So what is the catch?</u>

We use 'countable additivity axiom'
However, is it a correct application?
Do we have a sequence of countable disjoint set?
Does this sequence of set really capture the real line between [0,1]?

- Integers are ‘countable infinite set’
- Continuous set is a BIGGER set: uncountable
- Infinity has different sizes!
- Countable additivity axiom is not appropriate here

How do we work out these issues?

We are still 2 chapters ahead!

But moral of the story (especially true for continuous model)

Zero probability does not mean impossible

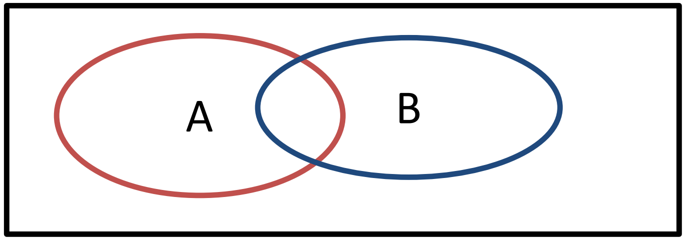
Probabilistic problem solving

- Specify sample space
- Define probabilistic law
- Identify event of interest
- Calculate probability

P1. Give a mathematical derivation of the formula,

$$P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A) + P(B) - 2P(A \cap B)$$

Since the set $(A \cap B^c)$ and the set $(A^c \cap B)$ are disjoint



By taking the union of two disjoint set,
We can invoke Axiom 3 (Additivity axiom)

$$P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A \cap B^c) + P(A^c \cap B)$$

Since $A = (A \cap B) \cup (A \cap B^c)$: These two are disjoint set too!

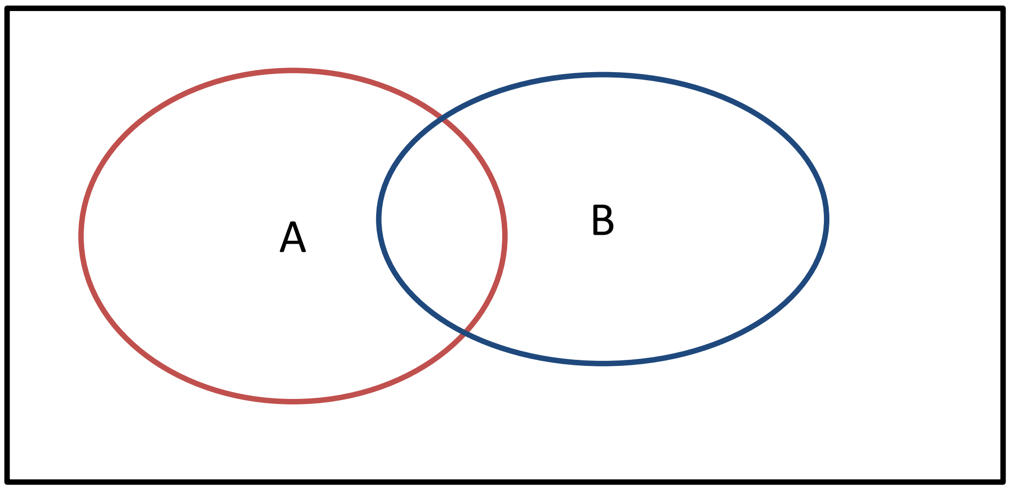
$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

Similar logic: $P(A^c \cap B) = P(B) - P(A \cap B)$

Combining everything above:

$$P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A \cap B^c) + P(A^c \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$$



Visualizing the drawing is the easiest!

P2. Problem 1.5 in text

Geniuses and chocolates

Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover

Let's start by writing down some information:

A: The event that a randomly selected student is a genius
B: The event that a randomly selected student is a chocolate lover

Lets start with some algebra, and probability law:

$$1 = P(A \cup B) + P((A \cup B)^c)$$

Next,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Next,

$$P((A \cup B)^c = P(A^c \cap B^c)$$

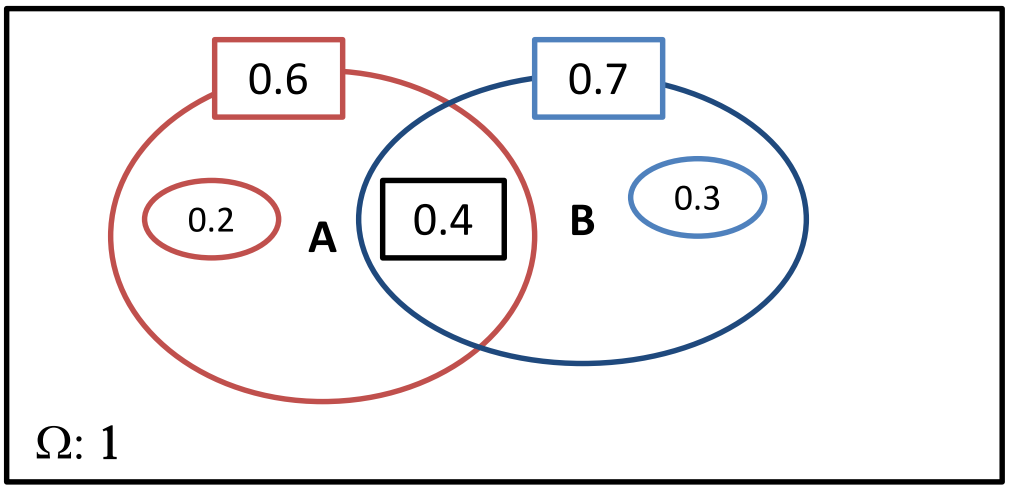
Back,

$$1 = P(A) + P(B) - P(A \cap B) + P(A^c \cap B^c) = 0.6 + 0.7 - 0.4 + P(A^c \cap B^c)$$

Finally,

$$P(A^c \cap B^c) = 0.1$$

Using Venn diagram is also quite simple



P3:

A six-sided die is loaded in a way that each even face is twice as likely as each odd face. Construct a probabilistic model for a single roll of this die, find the probability that a 1,2, or 3 will come up

Let c denote the probability of the event of a single odd face

Then, the probability of a single even face is $2c$

Adding all possible outcomes in this experiment:

3 odd faces + 3 even faces = $3c + 6c = 9c$

$$P(\text{sample space}) = 1$$

$$9c = 1, c = 1/9$$

Answer?

$$P\{(1,2,3)\} = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = 4/9$$

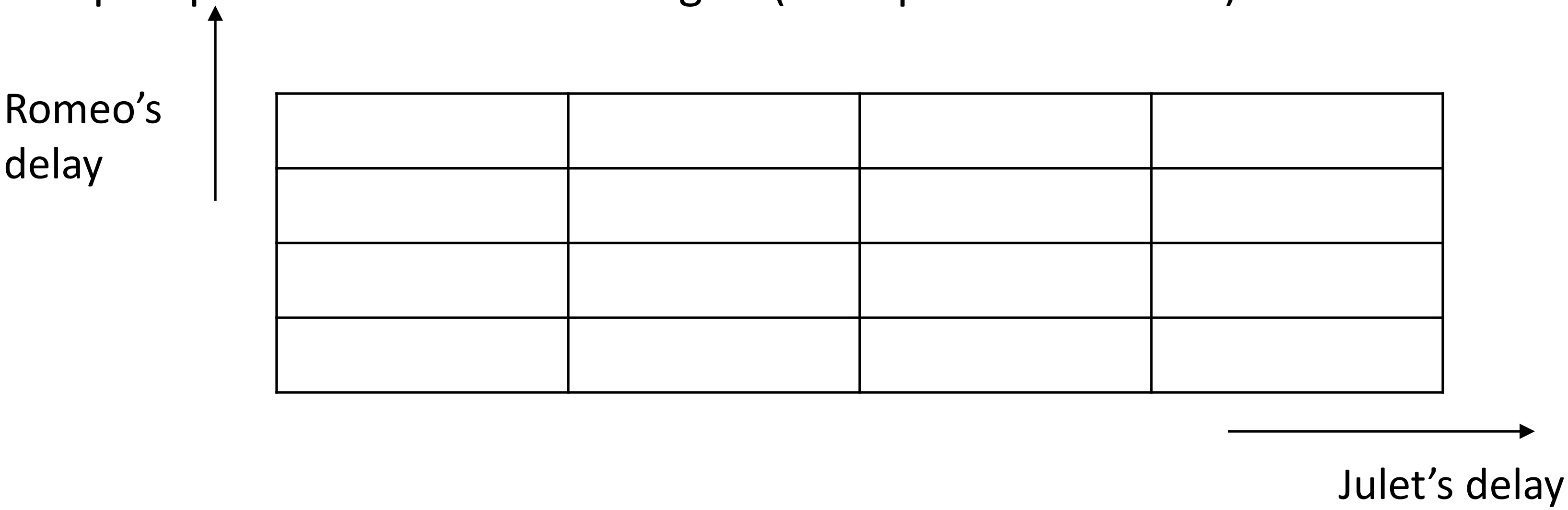
P4. Example 1.5 page 13

Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

Let’s set up a sample space with a <u>simpler</u> problem to gain intuition

Romeo and Juliet each can only arrive in 15 minutes increment

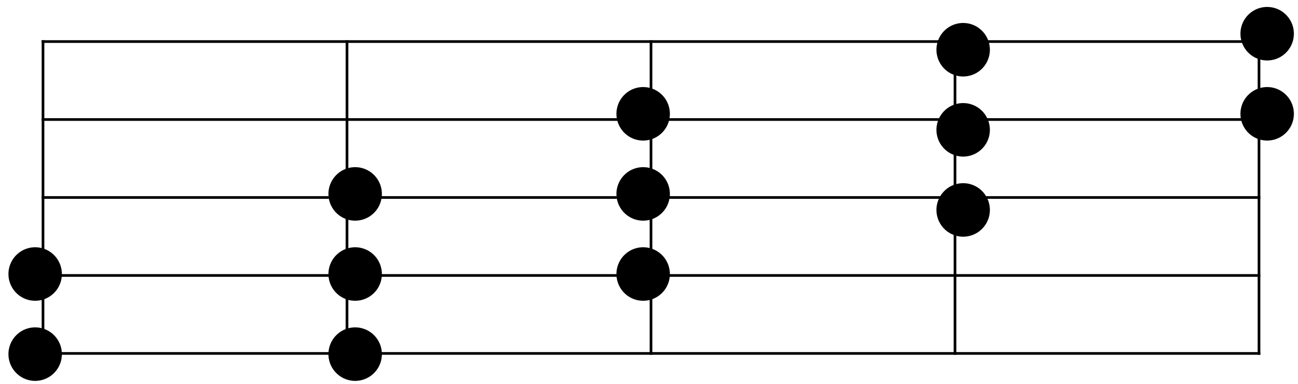
Sample space? Let’s draw it as a grid (each point is 15 mins)



All pair of points are equally-likely, 25 points total on the grid
Each point has probability 1/25
Discrete uniform law!

What is the probability that Romeo and Juliet will meet up for their date under this case?

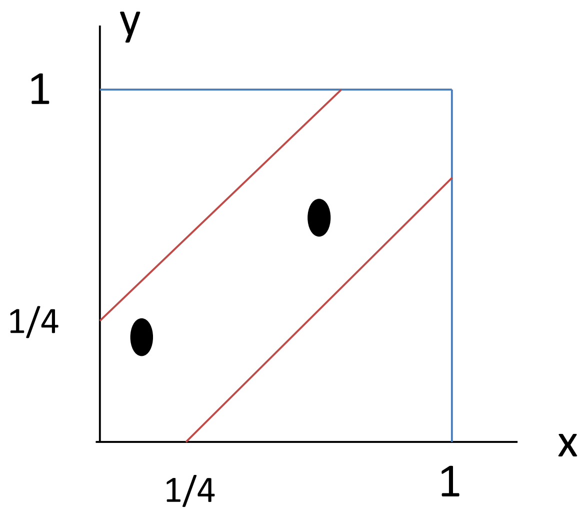
Essentially, count how many points which of these 25 outcomes result in both people arriving within 15 minutes of one another



So there are 13 points, the probability that they will meet up is 13/25

Back to the original problem,

Now instead of discrete arrival time, it is continuous, the sample space is essentially a unit (hour) square



Probability law?

Area

Now, identify the events of interest, which is Romeo and Juliet arrive within 15 minutes of each other

Now, it's just a matter of area computation,
Square size =1; 1 – 2*(3/4*3/4*1/2) = 7/16