國立清華大學
NATIONAL TSING HUA UNIVERSITY

EE 306001
Probability

Lecture 19: further topics on random variable

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- From the above two equation (first – second)

$$X - E[X] = -(Y - E[Y])$$

First calculate the covariance:

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= -E\left[(X - E[X])^2\right] = -var(X)$$

Now compute correlation coefficient

$$\rho = \frac{cov(X, Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{-var(X)}{\sqrt{var(X)var(X)}} = -1$$

Can we prove that $-1 \leq \rho \leq 1$?

$$\rho = \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{-var(X)}{\sqrt{var(X)var(X)}} = -1$$

- First we have to know a very important inequality
  - Schwarz inequality, for any two random variables X,Y

$$(E[XY])^2 \leq E[X^2]E[Y^2]$$

Now:

$$0 \leq E\left[\left(X - \frac{E[XY]}{E[Y^2]}Y\right)^2\right]$$

$$= E\left[X^2 - 2\frac{E[XY]}{E[Y^2]}XY + \frac{(E[XY])^2}{(E[Y^2])^2}[Y^2]\right] =$$

$$= E[X^2] - \frac{(E[XY])^2}{E[Y^2]}E\left[\frac{2}{E[XY]}XY - \frac{Y^2}{E[Y^2]}\right] = E[X^2] - \frac{(E[XY])^2}{E[Y^2]}E[2-1]$$

$$= E[X^2] - \frac{(E[XY])^2}{E[Y^2]}$$

Sum of random number of independent random variables

Over the weekend, you are going to visit a random number of bookstores, at each store, you are going to spend a random amount of money

Let N be number of stores that you are visiting, n is an integer (non-negative)

Each time you walk into a store, your mind is refreshed, and you just buy a random number of books that has nothing to do with what you have done for the day, each time you enter a book store as a brand new person, buys a random number of books, and spend a random amount of money

- Now let X_i be the money spent in store i
  - X_i assume i.i.d.
  - Independent of N

- Now let's set Y be the total money spent on book
  - $Y = X_1 + X_2 + X_3 \ldots + X_N$
  - We are dealing with sum of random variable except the N itself is also a random variable
  - First let's compute $E[Y]$?
  - Let's work in the conditional universe
    - Say if we are given $N = n$

$$E[Y|N = n] = E[X_1 + X_2 + X_3 \ldots + X_n|N = n]$$
$$= E[X_1 + X_2 + X_3 \ldots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= nE[X]$$

If we don't know N before hand,

$$E[Y|N] = NE[X]$$

- This is a random variable, and if you are given N to a specific value, then you get a number!

- Now we can invoke the iterated expectation law

$$E[Y] = E\big[E[Y|N]\big] = E\big[NE[X]\big] = E[X]E[N]$$

  - $E[X]$ is a number

  - This should also be intuitively easy!

- What if I want to know the variance in this case?

$$var(Y) = E[var(Y|N)] + var(E[Y|N])$$

  - $var(E[Y|N]) = var(NE[X]) = E[X]^2 var(N)$
    - Recall $var(aX) = a^2 var(X)$
    - Variability in how much money you are spending as the randomness exists in how many stores you visit
  - $var(Y|N = n) = nvar(X)$
  - $var(Y|N) = Nvar(X)$
  - $E[var(Y|N)] = E[Nvar(X)] = var(X)E[N]$
    - Randomness exists inside each store

So the total variability exists in how much you are going to spend

$$var(Y) = E[N]var(X) + (E[X])^2 var(N)$$

New topic

- Transforms of random variable
  - r.v.'s are functions, transform are a different representation of a functions, imagine, Fourier transform
  - Intuition around transforms in probability is kinda abstract, but often quite useful for mathematical manipulation

- Definition:
  - For a random variable X, the transform (or something called moment generating function) is defined below:

$$M_X(s) = E\left[e^{sX}\right]$$

  - s is a scalar parameter

- Let's write out the actual formula:

$$M(s) = \sum_{x} e^{sX} p_X(x)$$

$$M(s) = \int_{-\infty}^{\infty} e^{sX} f_X(x) dx$$

- Note that these transforms are not numbers, they are still a function of parameter s
- Linear function of X (e.g., $Y = aX + b$)

$$M_Y(s) = E\left[e^{s(aX+b)}\right] = e^{sb} E\left[e^{saX}\right] = e^{sb} M_X(sa)$$

Sample transforms

- Poisson random variable with parameter λ:

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,3 \ldots$$

Now, let's try to transform it:

$$M(s) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!}$$

Now, we can set $a = e^s \lambda$

$$M(s) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{a^x}{x!} = e^{-\lambda} e^{a} = e^{a-\lambda}$$

$$= e^{\lambda(e^s - 1)}$$

Sample transforms

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$$= e^{\lambda(e^s - 1)}$$

Geometric random variable

$$f_X = \lambda e^{-\lambda x}, x \geq 0$$

$$M(s) = \lambda \int_0^\infty e^{sx} e^{-\lambda x} dx$$
$$= \lambda \int_0^\infty e^{(s-\lambda)x} dx$$
$$= \lambda \frac{e^{(s-\lambda)x}}{s-\lambda} |_0^\infty$$
$$= \frac{\lambda}{\lambda - s}$$

Note, this is only defined if $s < \lambda$

Now why is this called moment-generating function?

- You can compute the moment of a random variable once you get the transform, how?

$$M(s) = \int_{-\infty}^{\infty} e^{sX} f_X(x)dx$$

Let’s take the derivative with respect to the s

$$\frac{d}{ds}M(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{sx} f_X(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds} e^{sx} f_X(x)dx = \int_{-\infty}^{\infty} x e^{sx} f_X(x)dx$$

Now, if we are looking for first moment ($E[X]$)

$$\frac{d}{ds}M(s)|_{s=0} = \int_{-\infty}^{\infty} x f_X(x)dx$$

- In general, to find the n^th moment

$$\frac{d^n}{ds^n} M(s)\Big|_{s=0} = \int_{-\infty}^{\infty} x^n f_X(x) dx = E[X^n]$$

- Let's do a quick example (Geometric random variable)

$$f_X = \lambda e^{-\lambda x}, x \geq 0$$

$$M(s) = \frac{\lambda}{\lambda - s}$$

Lets try to find $E[X]$

$$\frac{d}{ds} M(s) = \frac{\lambda}{(\lambda - s)^2}$$

Evaluate that at $s = 0$

$$= \frac{1}{\lambda} = E[X]$$

- How about second moment?

$$\frac{d^2}{ds^2}M(s) = \frac{2\lambda}{(\lambda - s)^3}$$

Evaluating at $s = 0$

$$E[X^2] = \frac{2}{\lambda^2}$$

We can then use this to find variance :

$$var(X) = E[X^2] - (E[X])^2$$

Inversion property of transform

- An important property of transform is that $M_X(s)$can be inverted to get back to the original probability law (e.g., $f_X(x)$ – like fourier transform

- Some appropriate conditions need to be matched, however, we can safely assume they are all satisfied

- A detailed understanding is beyond the scope of this course

Simple example of this property

Assume we are told the transform for a random variable X is of the following form:

$$M(s) = \frac{pe^s}{1-(1-p)e^s}$$

where $0 < p \leq 1$

We want to find the PDF of X?

- First let's recall geometric series is achieved with this infinite sum:

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots$$

Which is valid for $|\alpha| < 1$

Now, let's set $\alpha = (1-p)e^s$, and for s close to zero to make $(1-p)e^s < 1$ (convergence criterion)

We have:

$$M(s) = \frac{pe^s}{1-(1-p)e^s} = pe^s\left(1 + (1-p)e^s + (1-p)^2e^{2s} + \cdots\right)$$

We can then infer (find similar form) back, this is just a geometric distribution in the transform space:

$$P(X = k) = p(1-p)^{k-1}$$

$$M(s) = \sum e^{sX} p_X(x)$$

- Quick exercise:
- $M(s) = \frac{pe^s}{1-(1-p)e^s}$

Let's try to find $E[X]$

$$\frac{d}{ds}M(s) = \frac{pe^s}{1-(1-p)e^s} + \frac{(1-p)pe^{2s}}{(1-(1-p)e^s)^2}$$

Setting $s = 0$

$$E[X] = 1/p$$

Sums of independent random variables

A particular convenient way to look at sums of independent random variable is to look at its transform

Transforms makes the summation results in ‘*multiplication*’ instead of ‘*convolution*’

Much easier to work with and analyze it

Let $Z = X + Y$

$$M_z(z) = E\left[e^{sZ}\right] = E\left[e^{s(X+Y)}\right] = E\left[e^{sX}e^{sY}\right]$$

Since, X and Y are independent, e^{sX}, e^{sY} are also independent

Then expectation factors:

$$M_z(z) = E\left[e^{sX}e^{sY}\right] = E\left[e^{sX}\right]E\left[e^{sY}\right] = M_X(x)M_Y(y)$$

This can be generalized if you have *n* independent variables

$$M_z(z) = M_{X_1}(x)M_{X_2}(x)M_{X_3}(x) \ldots M_{X_n}(x)$$

Quick example

- If we have two variables X, Y both Poisson distributed
  - Each with mean λ and μ

We can get the transform functions of each as follows:

$$M_X(s) = e^{\lambda(e^s-1)}, M_Y(s) = e^{\mu(e^s-1)}$$

If they are independent,

$$M_Z(s) = M_X(s)M_Y(s) = e^{\lambda(e^s-1)}e^{\mu(e^s-1)}$$
$$= e^{(\lambda+\mu)(e^s-1)}$$

We can tell from this transform function:

Z is a Poisson distribution with mean $(\lambda + \mu)$: sum of Poisson

Another quick example

Let X and Y be independent normal random variables with means μ_X, μ_Y, and variances σ_X^2, σ_Y^2, now let $Z = X + Y$

$$M_X(s) = \exp\left\{\frac{\sigma_X^2 s^2}{2} + \mu_X s\right\}, M_Y(s) = \exp\left\{\frac{\sigma_Y^2 s^2}{2} + \mu_Y s\right\},$$

So Z is the following form:

$$M_Z(s) = \exp\left\{\frac{(\sigma_X^2 + \sigma_Y^2)s^2}{2} + (\mu_X + \mu_Y)s\right\}$$

So, Z is also a normal distribution with mean $(\mu_X + \mu_Y)$ and variance $\sigma_X^2 + \sigma_Y^2$ (sum of normal distribution)

Sum of a random number of independent random variables

- Say we have $Y = X_1 + X_2 + \cdots + X_N$
  - N is a random variable that takes non-negative integer values
  - X_1, X_2, X_3… are iids variables
- First:

$$E[Y|N] = NE[X]$$
$$E[Y] = E\big[E[Y|N]\big] = E[X]E[N]$$

- Also:

$$var(Y) = E[var(Y|N)] + var\big[E[Y|N]\big]$$
$$= E[Nvar(X)] + var(NE[X])$$
$$= E[N]var(X) + (E[X])^2 var(N)$$

What about this in terms of moments?

$$M(s) = \sum_{x} e^{sX} p_X(x)$$

$$M(s) = \int_{-\infty}^{\infty} e^{sX} f_X(x) dx$$

- Let Y $= X + X1$

$$M_Y(y) = E\left[e^{sX} e^{sX1}\right] = E\left[e^{sX}\right] E\left[e^{sX1}\right] = M_X(x) M_{X1}(x1)$$

- Now,

$$E\left[e^{sY} | N = n\right] = E\left[e^{sX_1} e^{sX_2} \ldots e^{sX_N} | N = n\right]$$

$$= E\left[e^{sX_1} e^{sX_2} \ldots e^{sX_n}\right]$$

$$= E\left[e^{sX_1}\right] E\left[e^{sX_2}\right] \ldots E\left[e^{sX_N}\right]$$

$$= \left(M_X(s)\right)^n$$

- This is transform of Y conditioned on $N = n$
  - Use iterated expectation to find the transform

$$M_Y(s) = E\left[e^{sY}\right] = E\left[E\left[e^{sY} | N\right]\right] = E\left[(M_X(s))^N\right]$$

$$= \sum_{n=0}^{\infty} \left(M_X(s)\right)^n p_N(n)$$

- Now let's look at it closer:

$$\left(M_X(s)\right)^n = e^{\log\left(M_X(s)\right)^n} = e^{n \log M_X(s)}$$

$$M_Y(s) = \sum_{n=0}^{\infty} \left(M_X(s)\right)^n p_N(n) = \sum_{n=0}^{\infty} e^{n \log M_X(s)} p_N(n)$$

- Now let's look at:

$$M_N(s) = E\left[e^{sN}\right] = \sum_{n=0}^{\infty} e^{sn} p_N(n)$$

- Comparing these two:

$$M_Y(s) = M_N(\log M_X(s))$$

Example

Jane visits a number of bookstores, any given bookstore carries the book with probability p, independent of others.

In a typical bookstore visited, Jane spends a random amount of time, exponentially distributed with parameter, λ, until she either find the book or decide that the bookstore does to have it.

We assume that Jane will keep visiting bookstores until she buys the book, the time she spend at each time is independent of others. We wish to find the mean, variance, and PDF of the total time spent in bookstores

- The total number N of bookstore visited in geometrically-distributed with parameter p

- The total time, Y, spent in bookstores is the sum of N exponentially-distributed variables $(X_1, X_2, \dots, X_N)$

$$E[Y] = E\big[E[Y|N]\big] = E[X]E[N] = \frac{1}{\lambda}\left(\frac{1}{p}\right)$$

$$var(Y) = E[N]var(X) + (E[X])^2 var(N)$$
$$= \frac{1}{p} * \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\frac{1-p}{p^2} = \frac{1}{\lambda^2 p^2}$$

- We have the mean and the variance of Y
- How about the PDF?
  - We can do it directly from the transforms

- First:
  - Geometric PDF's transform: $M_N(s) = \frac{pe^s}{1-(1-p)e^s}$
  - Exponential PDF's transform: $M_X(s) = \frac{\lambda}{\lambda - s}$

How do we find $M_Y(s)$?

$$M_Y(s) = M_N(\log M_X(s))$$

- Start with $M_N(s)$ then replace $s = \log M_X(s)$ or $e^s = M_X(s)$

$$M_Y(s) = \frac{pM_X(s)}{1-(1-p)M_X(s)} = \frac{\frac{p\lambda}{\lambda - s}}{1-(1-p)\frac{\lambda}{\lambda - s}}$$

Simplification:

$$M_Y(s) = \frac{p\lambda}{p\lambda - s}$$

What is the PDF associated with this transform?

– That is exponential distribution with parameter of $p\lambda$

$$f_Y(y) = p\lambda e^{-p\lambda y}, y \geq 0$$

– Interesting results, if you add 'fixed' (known) exponential does not get you an exponential distribution, say you add 2 exponentials:

$$M_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$$

This is not exponentials , if you know the exact, and if you don't know the exact makes a different

Review of the sections

- Derived distribution:
  - To calculate the PDF of a function $Y = g(X)$
    - First: calculate the CDF of Y

$$F_Y(y) = P(g(X) \leq y) = \int_{\{x|g(x)\leq y\}} f_X(x)dx$$

    - Differentiate with respect to CDF to get PDF

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

  - This is the standard two-step approach

- Suppose that the function g is a monotonic and that for some function h, and all x in the range of X we have:

$$y = g(x) \text{ if and only if } x = h(y)$$

- Under this condition, assume that h is differentiable

$$f_Y(y) = f_X\big(h(y)\big)\left|\frac{dh}{dy}(y)\right|$$

- Covariance and Correlation
  - The covariance of X and Y is defined as follow:

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

  - If two variables are ‘uncorrelated’, that means $cov(X,Y) = 0$
  - X, Y independent means uncorrelated, converse is not true

- $var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$
- Correlation coefficients (unit-less measurement of linear relationship)

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}$$

and it satisfies:

$$-1 \leq \rho(X,Y) \leq 1$$

- Conditional expectation and variance
  - Law of iterated Expectations

$$E[X] = E\big[E[X|Y]\big]$$

  - Law of total variance

$$var(X) = E[var(X|Y)] + var(E[X|Y])$$

- Transforms
  - The transform associated with a random variable X is given by

$$M_X(s) = E\left[e^{sX}\right] = \begin{cases} \sum_X e^{sx} p_X(x) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x) dx \end{cases}$$

  - The distribution of a random variable is completely determined by the corresponding transform
  - Moment generating function

$$M_X(0) = 1, \frac{d}{ds} M_X(s) \bigg|_{s=0} = E[X], \frac{d^n}{ds^n} M_X(s) \bigg|_{s=0} = E[X^n]$$

  - If $Y = aX + b$, then $M_Y(s) = e^{sb} M_X(as)$
  - If X and Y are independent, then $M_{X+Y}(s) = M_X(s) M_Y(s)$

Example

a)

You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained

- Let X_i be independent Bernoulli random variable
  - 1 when the coin flipped results in head

- Let N be the number of flips

- $E[X_i] = \frac{1}{2}$
- $var(X_i) = \frac{1}{4}$

- $E[N] = \frac{7}{2}$: $(1 + 2 + 3 + 4 + 5 + 6) * \frac{1}{6}$
- $var(N) = \frac{35}{12}$: $(n^2 - 1) * \frac{1}{12}$ - discrete uniform distribution

- Now we have all the information available:

We want to compute the expected number of heads (Y is sum of $X_i's$)

$$E[Y] = E[X_i]E[N] = \frac{1}{2} * \frac{7}{2} = \frac{7}{4}$$

$$var(Y) = var(X_i)E[N] + (E[X_i])^2 var(N)$$
$$= \frac{1}{4} * \frac{7}{2} + \frac{1}{4} * \frac{35}{12} = \frac{77}{48}$$

- Repeat part (a) for the case when you roll two dice

How do we proceed?

Imagine you do it twice, with each experiment independent of one another,

So what should the expectation to be (a summation)?

$$\frac{7}{2}$$

So what should be the variance to be (also a summation)?

$$\frac{77}{24}$$

Example

Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i

a) Are X_1, X_2 uncorrelated, positively correlated, negative correlated? Give an intuitive answer

They should negatively correlated, if you are getting a large number of 1's you are of course getting a fewer number of 2's

b) Compute the covariance $cov(X_1, X_2)$

$$cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

Now let's define some variable:

- A_t: Bernoulli random variable that is 1 when the dice shows 1 at t^{th} toss
- B_t: Bernoulli random variable that is 1 when the dice shows 2 at t^{th} toss

So,

$$X_1 = A_1 + \cdots + A_t + \cdots + A_n$$
$$X_2 = B_1 + \cdots + B_t + \cdots + B_n$$

First,

$$E[A_t B_t] = 0$$

Why, because if one of them is 1, the other has to be 0

Then,

$$E[A_t B_s] = E[A_t]E[B_s] = \frac{1}{\mathrm{k}} * \frac{1}{\mathrm{k}}, \text{for s} \neq t$$

Now,

$$E[X_1 X_2]$$
$$= E([A_1 + \cdots + A_t + \cdots + A_n])([B_1 + \cdots + B_t + \cdots + B_n)]$$
$$= nE[A_1(B_1 + \cdots + B_n)]$$
$$= n * (n-1) * \frac{1}{k} * \frac{1}{k}$$

- Now, we have everything needed to compute

$$cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$
$$= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}$$

As what we have argued in problem 1, X1, X2 are negatively correlated

Example

- A club has N members, where N is a random variable with PMF as follows:

$$p_N(n) = p^{n-1}(1-p), \text{for n=1,2,3} \ldots$$

On the second Tuesday night of every month, the club holds a meeting. Each member attends the meeting with probability q, independently of all the other members. If a member attends the meeting, he bring an amount of money, M, which is a continuous random variable:

$$f_M(m) = \lambda e^{-\lambda m}, m \geq 0$$

N, M, and whether the member attends the meeting are all independent

a) Find the expectation and variance of the number of members showing up at the meeting

Let's start by having all information at hand:

- K is the number of members attend the meeting
- B denotes whether a member would attend the meeting (Bernoulli)
- N is like Geometric (with success rate $(1-p)$)

$$E[N] = \frac{1}{1-p}, var(N) = \frac{p}{(1-p)^2}$$

$$E[M] = \frac{1}{\lambda}, var(M) = \frac{1}{\lambda^2}$$

$$E[B] = q, var(B) = q(1 - q)$$

- K is the number of members attend the meeting

$$K = B_1 + B_2 + \cdots + B_N$$

$$E[K] = E[N]E[B] = \frac{1}{1 - p} * q$$

$$var(K) = E[N]var(B) + (E[B])^2 var(N)$$

$$= \frac{q(1 - q)}{1 - p} + \frac{pq^2}{(1 - p)^2}$$

b) Find the expectation and variance for the total amount of money brought to the meeting

- Let G be the total money brought to the meeting

$$G = M_1 + \cdots + M_K$$

$$E[G] = E[M]E[K] = \frac{1}{\lambda} * \frac{q}{1-p}$$

$$var(G) = var(M)E[K] + (E[M])^2 var(K)$$
$$= \frac{q}{\lambda^2(1-p)} + \frac{1}{\lambda^2}\left(\frac{q(1-p)}{1-p} + \frac{pq^2}{(1-p)^2}\right)$$

Example

Romeo and Juliet have a date at a given time, and each, independently will be late by amounts of time X and Y, respectively, that are exponentially distributed with parameter λ

Find the PDF of $Z = X - Y$ using two step approach

- Let's break into 2 regions ($z \geq 0, z < 0$)

$$F_Z(z) = P(X - Y \leq z) = P(X \leq Y + z)$$

$$= \int_0^{\infty} \int_0^{y+z} f_{X,Y}(x, y) dx dy$$

$$= \int_0^{\infty} \lambda e^{-\lambda y} \int_0^{y+z} \lambda e^{-\lambda x} dx\, dy$$

$$= 1 - \frac{1}{2} e^{-\lambda z}, z \geq 0$$

- $z < 0$
- By symmetry
  - We know the distribution $Z = X - Y$ is the same as $-Z = Y - X$

$$F_Z(z) = P(Z \leq z) = P(-Z \geq -z) = P(Z \geq -z)$$
$$= 1 - F_Z(-z)$$

Put it together:

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2} e^{-\lambda z}, \text{if } z \geq 0 \\ \frac{1}{2} e^{\lambda z}, \text{if } z < 0 \end{cases}$$

- Simply differentiate it to get the PDF:

$$f_Z(z) = \begin{cases} \frac{\lambda}{2} e^{-\lambda z}, \text{if } z \geq 0 \\ \frac{\lambda}{2} e^{\lambda z}, \text{if } z < 0 \end{cases} = \left(\frac{\lambda}{2}\right) e^{-\lambda |z|}$$

This is called two-sided exponential PDF, also note as Laplace PDF

Example

How do we find the moment transform of Binomial?

Knowing that binomial distribution essentially is a summation of Bernoulli,

First, let’s find the moment transform of Bernoulli

$$M(s) = \sum_{x} e^{sX} p_X(x)$$

$$M_{X_i}(s) = (1-p)e^{0s} + pe^{1s} = 1 - p + pe^{s}$$

Summation of random variable?

Multiplication in transform,

Binomial is a sum of 'n' independent Bernoulli (parameter: n, p), expectation of joint factors, hence, moment function multiplies

$$M_Z(s) = (1 - p + pe^{s})^n$$

Example

We toss n times a biased coin whose probability of heads, denoted by q, is the value of a random variable Q, with mean μ and positive variance, σ^2

Let X_i be a Bernoulli rv. That models the outcome of the i^{th} toss (i.e., $X_i = 1$ if i^{th} toss is a head).

We assume that $X_1, \ldots, X_n$ are conditionally independent, given $Q = q$. Let X be the number of heads obtained in the n tosses

a) Find $E[X_i]$ and $E[X]$

From law of iterated expectations:

First:

$$E[X_i|Q] = Q$$

So,

$$E[X_i] = E\big[E[X_i|Q]\big] = E[Q] = \mu$$

$$E[X] = E[X_1] + \cdots + E[X_n] = n\mu$$

b) Find $cov(X_i, X_j)$, are X_i's independent of one another?

First for $i \neq j$

With conditional independent:

$$E[X_i X_j | Q] = E[X_i | Q] E[X_j | Q] = Q^2$$

Using law of iterated expectation:

$$E[X_i X_j] = E\left[E[X_i X_j | Q]\right] = E[Q^2]$$

Now we can compute covariance:

$$cov(X_i, X_j) = E[X_i X_j] - E[X_i] E[X_j] = E[Q^2] - \mu^2 = \sigma^2$$

We are told that σ^2 is strictly positive,

So $cov(X_i, X_j) \neq 0$, hence, they are not independent

For the case of $i = j$

First noting that $X_i^2 = X_i$

$$\begin{aligned} var(X_i) &= E\left[X_i^2\right] - (E[X_i])^2 \\ &= E[X_i] - (E[X_i])^2 \\ &= \mu - \mu^2 \end{aligned}$$