國立清華大學
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EE 306001
Probability

Lecture 15: continuous random variable

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Summary of joint PDF

Let X and Y be jointly continuous random variables with joint PDF $f_{X,Y}(x, y)$

- The joint PDF can be used to calculate probabilities

$$P\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

- The marginal PDFs of X and Y can be obtained from the joint PDF, using the following:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,, \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

- The joint CDF is defined by $F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$, and determines the joint PDF through the following formula:

$$f_{X,Y}(x, y) = \frac{d^2 F_{X,Y}}{dxdy}(x, y)$$

- A function $g(X, Y)$ of X and Y defines a new random variable, and:

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dxdy$$

  - If g is linear, of the form $aX + bY + c$, then we have:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Conditional PDF given an event

- The conditional PDF, $f_{X|A}$ of a continuous random variable X, given an event A with P(A)>0, satisfies

$$P(X \in B|A) = \int_B f_{X|A}(x)dx$$

- If A is a subset of the real line with $P(X \in A) > 0$, then

$$f_{X|\{X\in A\}}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in A)}, \text{ if } x \in A \\ 0, \end{cases}$$

- Let A1, A2, …, An be disjoint events that form a partition of the sample space, and assume that $P(A_i)>0$ for all *i*, then

$$f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)$$

- A version of total probability theorem

Conditional PDF given a random variable

Let X, Y be jointly continuous random variable with joint PDF $f_{X,Y}$

- The joint, marginal, and conditional PDF are related to one another via the following formula:

$$f_{X,Y}(x, y) = f_Y(y) f_{X|Y}(x|y)$$

The conditional PDF is defined only for those y's for which $f_Y(y) > 0$

Conditional Expectations

Let X and Y be jointly continuous random variables, and let A be an event with P(A) > 0

- Definition: the conditional expectation of X given the event A is defined by

$$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x) dx$$

- The conditional expectation of X given that Y=y is defined by

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|Y = y) dx$$

- The expected value rule: for a function g(X), we have:

$$E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$$

And

$$E[g(X)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

- Total expectation theorem: let A_1, A_2, …, A_n be disjoint events that form a partition of the sample space, and assume that $P(A_i)>0$ for all i, then

$$E[X] = \sum_{i=1}^{n} P(A_i) E[X|A_i]$$

Similarly,

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy$$

Independence

Let X and Y be jointly continuous random variables

- X and Y are independent if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y), \text{for all x, y}$$

- If X and Y are independent, then

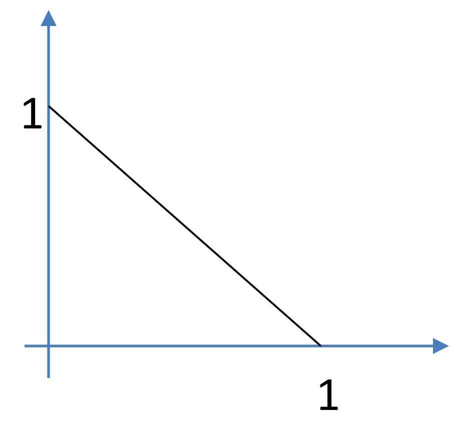
$$E[XY] = E[X]E[Y]$$

- If X and Y are independent, then

$$var(X + Y) = var(X) + var(Y)$$

Example 1

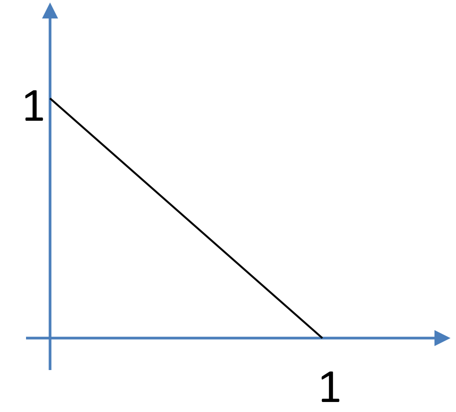
Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices (0,0), (0,1) and (1,0)



a) Find the joint PDF of X and Y

The area of the triangle is ½

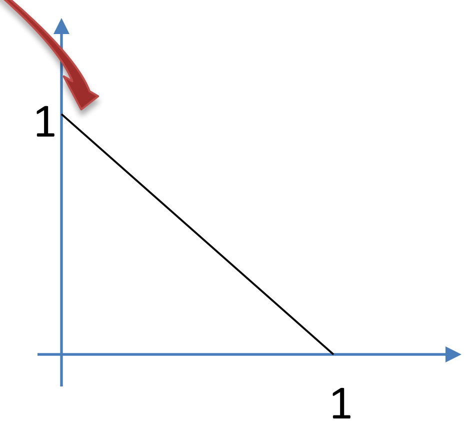
$$f_{X,Y}(x,y) = 2, (x,y) \text{ in the triangle shown}$$
$$= 0, \text{elsewhere}$$



b) Find the marginal PDF of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

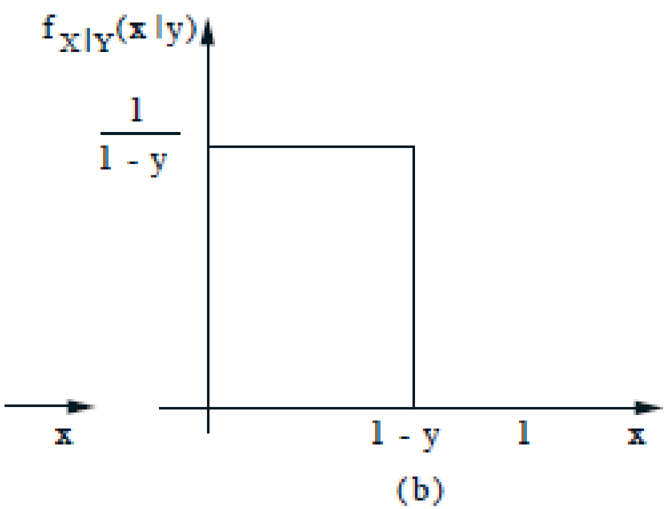
$$= \int_0^{1-y} 2\,dx = 2(1-y), 0 \leq y \leq 1$$



c) Find the conditional PDF of X given Y

- We have:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, 0 \le x \le 1-y$$



Intuitively, since the joint PDF is constant, the conditional PDF (a slice of joint PDF) is also constant.

Imagine the previous figure, if you take a slice of joint PDF at Y=y, X ranges from 0 to 1-y

In order for the condition PDF to integrate to 1, it's height must be 1/(1-y)

d) Find E[X|Y=y], and use the total expectation theorem to find E[X] in terms of E[Y]

For y >1 or y< 0, the conditional PDF is undefined

For 0 ≤ y ≤1, the conditional mean:

$$E[X|Y=y] = \int_0^{1-y} x \frac{1}{1-y} dx = \frac{1}{1-y}\frac{1}{2}x^2 \Big|_0^{1-y}$$

$$= \frac{1-y}{2}$$

- The total expectation theorem yields:

$$E[X] = \int E[X|Y=y] f_Y(y) dy = \int_0^1 \frac{1-y}{2} f_Y(y) dy$$

$$= \frac{1}{2} - \frac{1}{2} \int_0^1 y f_Y(y) dy = \frac{1-E[Y]}{2}$$

e) Use the symmetry of the problem to find the value of E[X]

- Because of symmetry:

We have $E[X] = E[Y]$

Therefore, $E[X] = \frac{(1-E[X])}{2}$

$$E[X] = \frac{1}{3}$$

Example 2

Random variables X and Y are distributed according to joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} ax, \text{if } 1 \le x \le y \le 2 \\ 0, \text{otherwise} \end{cases}$$

a) Evaluate the constant a

- Because of the required normalization property of any joint PDF,

$$1 = \int_{x=1}^{2} \int_{y=x}^{2} axdy\, dx = \int_{x=1}^{2} ax(2-x)dx$$
$$= a\left(2^2 - 1^2 - \frac{2^3}{3} + \frac{1^3}{3}\right) = \frac{2}{3}a$$

$$a = 3/2$$

b) Determine the marginal PDF $f_Y(y)$

- For $1 \leq y \leq 2$

$$f_Y(y) = \int_1^y ax\,dx = \frac{a}{2}\left(y^2 - 1\right)$$
$$= \frac{3}{4}\left(y^2 - 1\right)$$

And, $f_Y(y) = 0$, otherwise

$$f_{X,Y}(x,y) = \begin{cases} ax, \text{if } 1 \leq x \leq y \leq 2 \\ 0, \text{otherwise} \end{cases}$$

c) Determine the expected value of 1/x, given that Y=3/2

- Since we know $Y = 3/2$, then, for $1 \leq x \leq 3/2$

$$f_{X|Y}\left(x|Y = \frac{3}{2}\right) = \frac{f_{X,Y}\left(x, \frac{3}{2}\right)}{f_Y\left(\frac{3}{2}\right)}$$

$$= \frac{(3/2)x}{\frac{3}{4}\left(\left(\frac{3}{2}\right)^2 - 1^2\right)} = \frac{8x}{5}$$

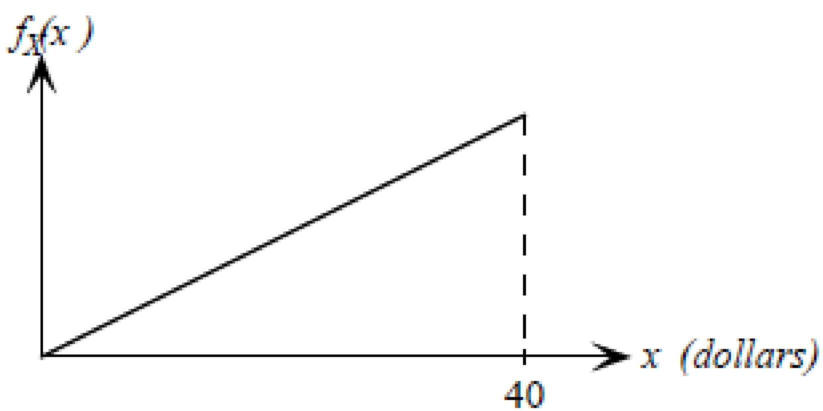
- Knowing this:

$$E\left[\frac{1}{X} | Y = \frac{3}{2}\right] = \int_{1}^{\frac{3}{2}} \frac{1}{x} \frac{8x}{5} dx = \frac{4}{5}$$

Example 3

Paul is vacationing. The amount X (in dollars) he takes to casino each evening is a random variable with the PDF shown in the figure.

At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in



a) Determine the joint PDF $f_{X,Y}(x, y)$

Multiplication rule:

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x)$$

What is $f_X(x)$?

$$f_X(x) = ax$$

$$1 = \int_0^{40} ax dx = 800a$$

So, $f_X(x) = x/800$

From the problem statement,

$$f_{Y|X}(y|x) = \frac{1}{2x}, \text{for } y \in [0, 2x]$$

- Therefore,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{1600}, \text{ if } 0 \le x \le 40, 0 < y < 2x \\ 0, \text{otherwise} \end{cases}$$

b) What is the probability that on any given night Paul makes a positive profit at the casino?

- Paul makes a positive profit if Y > X, the probability associated with that is:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{1600}, \text{ if } 0 \le x \le 40, 0 < y < 2x \\ 0, \text{otherwise} \end{cases}$$

$$P(Y > X) = \iint_{y>x} f_{X,Y}(x,y)dydx = \int_0^{40}\int_x^{2x} \frac{1}{1600} dydx$$
$$= \frac{1}{2}$$

c) Find the probability density function of Paul's profit on any particular night, $Z = Y - X$. What is $E[Z]$?

Let's compute the joint density first (easier to get)

$$f_{Z,X}(z,x) = f_X(x) f_{Z|X}(z|x)$$

Since Z is conditionally uniformly distributed given X,

$$f_{Z|X}(z|x) = \frac{1}{2x}, \text{for } -x \le z \le x$$

Therefore,

$$f_{X,Z}(x,z) = \frac{1}{1600}, \text{for } 0 \le x \le 40, -x \le z \le x \ (z \le |x|)$$

Now, we can compute marginal density $f_Z(z)$

$$f_Z(z) = \int_x f_{X,Z}(x,z)dx = \int_{|z|}^{40} \frac{1}{1600} dx$$

$$= \begin{cases} \dfrac{40-|z|}{1600}, \text{if } |z|<40 \\ 0, \text{otherwise} \end{cases}$$

Bayes rule

- X is unknown, but we have some prior belief in how X is distributed
- We observe random variable Y
- Need a model of the box:
  - If the true state of the world is X, how do we expect Y to be distributed
  - Inference problem, knowing Y, what can we say about X?
  - Inference problem is about finding probability distribution
    - $P(X|Y)$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{P_{Y|X}(y|x)P_X(x)}{P_Y(y)}$$

$$P_Y(y) = \sum_x p_X(x)p_{Y|X}(y|x)$$

- Say, this is the discrete case
- Typical example:
  - $X = 1,0$: airplane present/no present
  - $Y = 1,0$: something did/did not register on radar
- Inference problem:
  - $P(X|Y)$: given the radar measurement, calculate the probability that the plane is up there

- Continuous case:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_Y(y) = \int_x f_X(x)\, f_{Y|X}(y|x) dx$$

- Typical example
  - X: some signal, e.g., current through resistor, with $f_X(x)$
  - Y: signal with some noise, e.g., Gaussian noise
- Inference problem is the same
  - Compute $f_{X|Y}$
  - $f_{Y|X}$ is the model of signal with noise

Discrete X, Continuous Y

- Typical example (communication)
  – X: discrete signal (say a bit, 0,1),
  – Y: measured signal (X + Gaussian noise)

- prior: $P_X(x)$
  - Could be equally-likely to send 0,1 (PMF)

– Continuous noise model: $f_{Y|X}(y|x)$
  - Conditional densities of y in a universe that's specified by a particular value of x

$$P(X = x|Y = y) \approx P(X = x|y \leq Y \leq y + \delta)$$

$$= \frac{P(X = x)P(y \leq Y \leq y + \delta|X = x)}{P(y \leq Y \leq y + \delta)}$$

$$\approx \frac{P(X = x)f_{Y|X}(y|x)\delta}{f_Y(y)\delta}$$

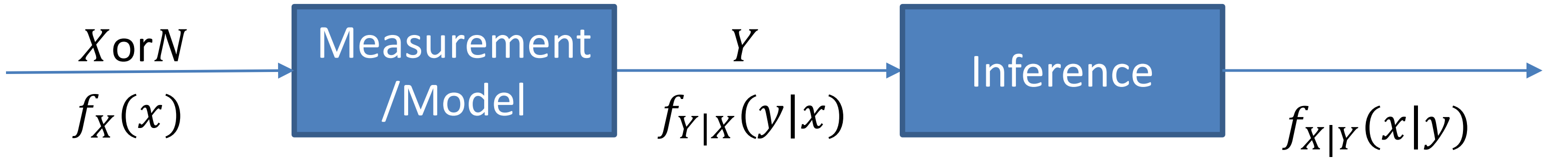
$$= \frac{P_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_Y(y) = \sum_i P_X(i)f_{Y|X}(y|i)$$

Put it together:

$$P_{X|Y}(x|y) = \frac{p_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$

Look at it again: Bayes rule



- If Y is a continuous random variable, we have:
  - If X is continuous random variable

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

So,

$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t) dt}$$

- If N is discrete random variable, we have

$$f_Y(y) P(N = n|Y = y) = p_N(n) f_{Y|N}(y|n)$$

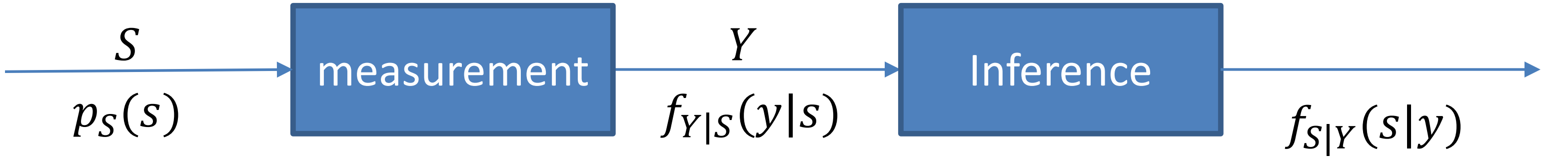
Resulting in the formula:

$$P(N = n|Y = y) = \frac{p_N(n) f_{Y|N}(y|n)}{f_Y(y)} = \frac{p_N(n) f_{Y|N}(y|n)}{\sum_i p_N(i) f_{Y|N}(y|i)}$$

A quick example

- A Binary signal S is transmitted, and we are given that
$$P(S = 1) = p, P(S = -1) = 1 - p$$

The received signal is $Y = N + S$, where N is Gaussian noise (μ=0, σ^2=1), independent of S. What is the probability that S=1, as a function of the observed value y of Y?



$$P(S=1|Y=y)=\frac{p_S(1)f_{Y|S}(y|1)}{f_Y(y)}$$

$$f_Y(y)=\sum_i p_N(i)f_{Y|N}(y|i)$$

What is $f_{Y|S}$?

If the measurement is additive Gaussian noise ($N(0,1)$) to the original signal *S*, that is $Y=S+Noise$

$$f_{Y|S}(y|S=1)\sim N\left(\mu=1,\sigma^2=1\right)$$

$$f_{Y|S}(y|S=-1)\sim N(\mu=-1,\sigma^2=1)$$

$$f_{Y|S}(y|S=1) \sim N(\mu=1, \sigma^2=1) = \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}$$

$$f_{S|Y}(S=1|Y=y) = \frac{p_S(1) f_{Y|S}(y|1)}{f_Y(y)}$$

$$\frac{p * \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}}{\frac{p}{\sqrt{2\pi}} e^{-(y-1)^2/2} + \frac{1-p}{\sqrt{2\pi}} e^{-(y+1)^2/2}}$$

$$= \frac{pe^y}{pe^y + (1-p)e^{-y}}$$

With this probability density function, we can do ‘informed’ probabilistic inference, e.g., finding the probability that your received signal y is within a particular range

Continuous X, Discrete Y

$$f_{X|Y}(x|y) = \frac{f_X(x)p_{Y|X}(y|x)}{p_Y(y)}$$

$$p_Y(y) = \int_x f_X(x)p_{Y|X}(y|x)dx$$

- Typical example:
  - X: a continuous signal (intensity of light beam)
  - Y: discrete variable (count the number of photons)

Example 1

- A light bulb produced by the company is known to have an exponentially distributed lifetime Y. However, the company has been experiencing quality control problems. On any given day, the parameter λ of the PDF of Y is actually a random variable, uniformly distributed in the interval $[1,3/2]$.

- We test a light bulb and record its lifetime. What can we say about the underlying parameter λ

- Let's model the parameter, λ, as a uniform random variable Λ with PDF:

$$f_{\Lambda}(\lambda) = 2, \text{for } 1 \leq \lambda \leq \frac{3}{2}$$

So if we test the light bulb and record it lifetime $(Y = y)$, then how can we use that ($Y = y$) to modify our original belief of $f_{\Lambda}(\lambda)$

Therefore, the available information about Λ is captured by the conditional PDF $f_{\Lambda|Y}(\lambda|Y = y)$, we can compute it using continuous Bayes rule

$$f_{\Lambda|Y}(y) = \frac{f_{\Lambda}(\lambda) f_{Y|\Lambda}(y|\lambda)}{\int_{-\infty}^{\infty} f_{\Lambda}(t) f_{Y|\Lambda}(y|t)dt} = \frac{2\lambda e^{-\lambda y}}{\int_{1}^{3/2} 2te^{-ty}dt}, \text{for } 1 \leq \lambda \leq \frac{3}{2}$$

This is statistics – somewhat, we will come back to this

Think about it, after you measure it, what's the most likely parameter for this exponential distribution