10720 EECS 303003 Probability Quiz #3 Answer Sheet

Section A (25%)

1. (G1) Lifetime of pigs, dogs, and cats are independent and exponentially distributed. They have means lifetime of 10, 20,and 30 years respectively. If they born simultaneously. What is the probability of pig die first?

Solution:

X=1/10e-1/10x Y=1/20e-1/20y Z=1/30e-1/30z

P(x<y<z+x<z<y)=$\int_0^{infinite}\int_x^{infinite}\int_y^{infinite} XYZdzdydx+\int_0^{infinite}\int_x^{infinite}\int_z^{infinite} XYZdydzdx$=6/11

2. (G1) Let X be a continuous random variable with PDF given by

$$fx(x) = \frac{1}{2}e^{-|x|}, \text{ for all } x\in R.$$

If Y=X^2, find the CDF of Y.

Solution:

First, we note that $R_Y = [0, \infty)$. For $y \in [0, \infty)$, we have

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx \\ &= \int_0^{\sqrt{y}} e^{-x} dx \\ &= 1 - e^{-\sqrt{y}}. \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

3. (G4) Let X be a random number from (0,1). Find the probability density function of Y=1/X

Solution:

$$F_y(y) = p\left(\frac{1}{x} \le y\right) = p\left(x \ge \frac{1}{y}\right) = \frac{1-\frac{1}{y}}{1-0} = 1 - \frac{1}{y} \qquad ,y > 1$$

$$f_y(y) = \frac{d}{dy}F_y(y) = \frac{1}{y^2}$$

Therefore,

$$F_y(y) = \begin{cases} \frac{1}{y^2} & ,y > 1 \\ 0 & ,o.w. \end{cases}$$

4. (G5) Let X be uniformly distributed in the unit interval [0, 1]. Consider the random variable Y = g(X), where

$$g(x) = \begin{cases} 1, & if\ x \le \frac{1}{3} \\ 2, & if\ x > \frac{1}{3} \end{cases}$$

Find the expected value of Y by first deriving its PMF.

Solution:

The random variable Y = g(X) is discrete and its PMF is given by

$$p_Y(1) = \mathbf{P}(X \le 1/3) = 1/3, \qquad p_Y(2) = 1 - p_Y(1) = 2/3.$$

Thus,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\, dx = \int_0^{1/3} dx + \int_{1/3}^1 2\, dx = \frac{5}{3}.$$

5. (G6) An adult is a person whose age is 18-35. X follows an exponential distribution determining how long a person live in years. If we know the average life expectancy in Taiwan is 80 years, what is the probability that a person dies when they are in adult stage?

Solution:

$$P(18 \le X \le 35) = F(35) - F(18), \quad where\ E[X] = \frac{1}{\lambda}, \lambda = \frac{1}{80}$$

$$Since\ \ F_x(x) = 1 - \int_0^x \lambda^{e^{-\lambda x}}\ \ dx$$

$$F(35) = 1 - \int_0^{35} \frac{1}{80}^{e^{-\frac{1}{80}x}}\ \ dx\ = 0.3544$$

$$F(18) = 1 - \int_0^{18} \frac{1}{80}^{e^{-\frac{1}{80}x}}\ \ dx\ = 0.2015$$

Therefore,

$$P(18 \le X \le 35) = F(35) - F(18) = 0.1529$$

Section B (35%)

1. (G4) A point Y is randomly selected from [0, 2], and the other point X is randomly selected from [0, Y]. Find the probability density function of X.

Solution:

$$f(x,y) = f_{X|Y}(x|y)f_Y(y)$$

So

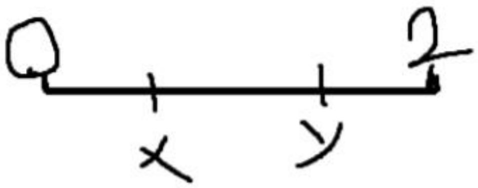
$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)dy$$

And

$$f_Y(y) = 1/2$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} \; for \;\; 0 < y < 2, 0 < x < y \\ 0 \; otherwise \end{cases}$$

Therefore, for $0 < x < 2$



$$f_X(x) = \int_x^2 \frac{1}{y}\frac{1}{2}\,dy = \frac{1}{2}ln\frac{2}{x}$$

$$f_X(x) = \begin{cases} ln\frac{1}{2x} & 0 < x < 2 \\ 0 & otherwise \end{cases}$$

2. (G4) The temperature at NTHU city is modeled as normal random variable with mean and standard deviation both equal to 30˚C (Celsius). A delicate bomb is designed to boom at 113˚F(Fahrenheit)(i.e. over 113˚F the bomb will explode). What is the probability that the bomb will explode? [Hint]: ˚C = $\frac{5}{9}$(˚F - 32)

Solution:

$$\mathrm{E[x]} = 30$$

and since we know the bomb will boom at 113˚F, probability that it will explode is

$$\mathrm{P(Y \le 113) = P(x \le 45) = P}\left(\mathrm{z} \le \frac{45 - E[x]}{\sigma_x}\right) = P\left(z \le \frac{15}{30}\right) = P(z \le 0.5) = \phi(0.5) \sim 0.6915$$

And so the answer =

$$1 - 0.6915 = 0.3085$$

3. (G5) Zwei is a naughty dog with his kennel located at the origin of a Cartesian coordinate system. His breeder, Ruby, finds out that the probability for finding Zwei at position (x,y) follows a joint probability density function:

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

We define the territory of a dog as a circular area where we could find that dog with probability 90%, and the center of the area is located at the kennel of that dog.
What is the radius R for the territory of Zwei? [Hint]: ln(10)=2.3

<u>Solution:</u>
First calculate the cumulative distribution function of radius R:

$$\mathrm{F(R)} = \int_0^R \int_0^{2\pi} \frac{1}{2\pi} e^{\frac{-r^2}{2}} r d\theta dr, \ \ r^2 = x^2 + y^2$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^R e^{\frac{-r^2}{2}} \ rdr = -e^{\frac{-r^2}{2}} |_0^{r=R} = 1 - e^{\frac{-R^2}{2}}$$

$$\text{Therefore, F(R)} = 1 - e^{\frac{-R^2}{2}} = 0.9$$

$$e^{\frac{-R^2}{2}} = 0.1, \quad \frac{-R^2}{2} = -\ln(10) = -2.3$$

$$R = \sqrt{4.6}$$

4. (G2) A company produces basketball, and the diameter of basketball X is a Laplacian random variable with probability density function

$$f_X(x) = \frac{1}{2\sigma} e^{-(|x-\mu|/\sigma)}, -\infty < x < \infty$$

where $\mu = 4, \sigma = 0.008$. A guess need a basketball which diameter is between the interval [4.00-0.016, 4.00+0.016]. The guess cannot accept diameter excess this range. Find the percentage the basketball the guess can accept of all basketballs. [Hint]: First calculate the $\mathrm{F(x)}|_{x \geq \mu}$.

<u>Solution:</u>

For $x \geq \mu$, you have

$$\begin{aligned}
F(x)\Big|_{x\geq\mu} &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\
&= \frac{1}{2b}\left\{\int_{-\infty}^{\mu} + \int_{\mu}^{x}\right\}\exp\left(-\frac{|t-\mu|}{b}\right)\mathrm{d}t \\
&= F(x)\Big|_{x=\mu} + \frac{1}{2b}\int_{\mu}^{x}\exp\left(-\frac{t-\mu}{b}\right)\mathrm{d}t \\
&= \frac{1}{2} - \frac{b}{2b}\left[\exp\left(-\frac{t-\mu}{b}\right)\right]_{\mu}^{x} \\
&= \frac{1}{2} - \frac{1}{2}\left(\exp\left(-\frac{x-\mu}{b}\right) - 1\right) \\
&= 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\{4.00 - 0.016 \leq X \leq 4.00 + 0.016\} \\
&= F(4.016) - F(3.984) \\
&= (1 - \frac{1}{2}e^{-\frac{4.016-4}{0.008}}) - \frac{1}{2}e^{+(3.984-4)/0.008} \\
&= 1 - \frac{1}{2}(e^{-2} + e^{2})
\end{aligned}$$

5. (G3) Suppose an engineer has designed a probe, its probability of locating its target is a random variable T with PDF

$$f_T(t) = \begin{cases} 1 + sin(2\pi t), if t \in [0,1], \\ 0, otherwise. \end{cases}$$

The random variable T is the time variable (seconds as units), meaning the probe only has at most 1 second to locate its target. In essence, the probe will have a fixed probability T = t of locating its target under specific conditions. Each search is an independent event.

[Hint]: $\int tsin(t)dt = \frac{sin(2\pi t) - 2\pi t cos(2\pi t)}{4\pi^2} + C)$

(a) Find the probability that the probe locates its target on the first search.

(b) Given the probe has located its target on its first search, find the conditional PDF of T.

Solution:

Let A be the event that the first search was successful. To calculate the probability P(A), we use the continuous version of the total probability theorem:

$$P(A) = \int_0^1 P(A \vee T = t) f_T(t)dt = \int_0^1 t\big(1 + sin(2\pi t)\big)dt$$

Therefore,

$$P(A) = \frac{\pi - 1}{2\pi}$$

Using Bayes rule,

$$f_{T\vee A}(t) = \frac{P(A|T=t)f_T(t)}{P(A)}$$

$$\begin{cases} \frac{2\pi t\left(1 + sin(2\pi t)\right)}{\pi - 1}, if 0 \le t \le 1, \\ 0, otherwise \end{cases}$$

Section C (40%)

1. Given the joint probability density function below,

$$f_{XY}(x,y) = \begin{cases} ce^{-2x}e^{-y}, & 0 \le y \le x \le \infty \\ 0, & elsewhere \end{cases}$$

(a) Find the constant c.
(b) Find the marginal pdfs of $f_X(x)$ and $f_Y(y)$.
(c) Find the variance of X.
(d) Find the probability $P(X + Y \le 2)$.

<u>Solution:</u>

(e) Find the constant c.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = c\int_0^{\infty}\int_0^{x} e^{-2x}e^{-y}dydx = c\int_0^{\infty} e^{-2x}(1 - e^{-x})dx = c\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$Therefore, \quad c = 6.$$

(f) Find the marginal pdfs of $f_X(x)$ and $f_Y(y)$.

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{x} 6e^{-2x}e^{-y}dy = 6e^{-2x}(1 - e^{-x}) = 6e^{-2x} - 6e^{-3x}, x \ge 0$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^{\infty} 6e^{-2x}e^{-y}dx = 6e^{-y}\frac{1}{2}e^{-2y} = 3e^{-3y}, y \ge 0$$

(g) Find the variance of X.

$$\mathrm{E[X]} = \int_{-\infty}^{\infty} xf_x(x)dx = \int_0^{\infty} x(6e^{-2x} - 6e^{-3x})dx = \frac{5}{6}$$

$$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)dx = \int_0^{\infty} x^2(6e^{-2x} - 6e^{-3x})dx = \frac{19}{18}$$

$$\mathrm{Var(X)} = \mathrm{E}[X^2] - \mathrm{E[X]}^2 = \frac{19}{18} - \left(\frac{5}{6}\right)^2 = \frac{13}{36}$$

(h) Find the probability $P(X + Y \le 2)$.

$$P(X + Y \le 2) = \int_0^{1}\int_y^{2-y} 6e^{-2x}e^{-y}dxdy = 6\int_0^{1} e^{-y}\frac{1}{2}(e^{-2y} - e^{-2(2-y)})dy$$

$$= 3\int_0^{1} e^{-3y} - e^{-4+y}dy = 1 - 4e^{-3} + 3e^{-4}$$