10720 EECS 303003 Probability Quiz #2 ANSWER

Easy

(G6)

Question:

We put 2 white balls and 8 black balls in an urn. A player picks successively 5 balls with replacement out of the urn. For each white ball picked, he wins 2 points and for each black ball picked, he loses 3 points.

We call X the discrete random variable representing the number of white balls picked and Y the discrete random variable representing the number of points of the player at the end of the game.

1. What is the PMF of X ? Find E[X] and V[X]
2. What is the PMF of Y ? Find E[Y] and V[Y]

Answer:

1) a. X is a binomial random variable with $n = 5$ and $p = \frac{1}{5}$

$X(\Omega) = [\![0,5]\!]$ and $\forall k \in [\![0,5]\!]$,

$$P(X=k) = \binom{5}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{5-k}$$

$$E[X] = np = 5 \times \frac{1}{5} = 1$$

$$Var[X] = np(1-p) = 5 \times \frac{1}{5} \times \frac{4}{5} = \frac{4}{5}$$

b. $Y = 2X - 3(5-X) = 5X - 15$

$Y(\Omega) = \{5k-15,\ k \in [\![0,5]\!]\} = \{-15,-10,-5,0,5,10\}$

$\forall k \in [\![0,5]\!]$,

$$P(Y = 5k-15) = P(X=k) = \binom{5}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{5-k}$$

$$E[Y] = E[5X-15] = 5E[X] - 15 = -10$$

$$Var[Y] = Var[5X-15] = 5^2 Var[X] = 20$$

(G3)

Question:

Recently, a seasonal flu outbreaks within Electrical Engineering Department. The probability for each person to get a flu is p; however, the students who has boy/girlfriend will double the probability of getting the flu. Supposed that there are 2m people in EE Department, and there are b couples (2b people), find the PMF, Expectation and Variance for all people in EE. (Don't need to simplify the answer)

Answer:

Let X = number of single people get flu.

Y = number of coupled people get flu.

Z = number of people get flu or not.

We can know that X,Y are independent, and Z=X+Y.

P(Z)=P(single)P(get flu|single)+P(coupled)P(get flu|coupled)=P(X)+P(Y)=$\frac{2m-2b}{2m} * p + \frac{2b}{2m} * 2p$

E(Z)=E(X)+E(Y)=2m*P(Z)=$(2m-2b) * p + 2b * 2p$

Var(Z)=Var(X)+Var(Y)={ $[1-E(X)]^2$*p+$[0-E(X)]^2$*(1-p) } + {$[1-E(Y)]^2$*2p+$[0-E(Y)]^2$*(1-2p) }

(G5)

Question:

Chen is a hard-working sophomore. Now he is facing 5 exams next week, but his time only allow him to prepare three of them. "Probability", "partial differential equation", "electromagnetism", "discrete math" and "signal and system" if he didn't prepare for the subject is always half chance to pass. After studying can he increase the chance to 0.6, 0.6, 0.7, 0.8, 0.9 respectively. By passing course can gain 3 credits each. And Chen chooses which subjects to read equally.

Given that he has given up studying "discrete math". And he has worked hard on "probability". what is the expectation he gets credits? (assume result only depend on this exam)

Answer:

Let X be the credit he gets

E[X] =

= 3 * ((0.6) + (1/3*0.5+2/3*0.6) + (1/3*0.5+2/3*0.7) + (0.5) + (1/3*0.5+2/3*0.9))

= 9.2

He can get 9.2 credits as expectation.

(G4)

Question:

Let X be a discrete random variable with the following PMF:

$$P_X(k) = \begin{cases} 0.2 fork = 1 \\ 0.2 fork = 2 \\ 0.5 fork = 3 \\ 0.1 fork = 4 \\ 0 otherwise \end{cases}$$

a. Find $E[X]$
b. Find $Var[X]$
c. If $Y = (X-2)^2$, find $E[Y]$

Answer:

a. $E[X] = 0.2 * 1 + 0.2 * 2 + 0.5 * 3 + 0.1 * 4 = 2.5$
b. $Var[X] = E[X^2] - (E[X])^2$

$$E[X^2] = 0.2 * 1^2 + 0.2 * 2^2 + 0.5 * 3^2 + 0.1 * 4^2 = 7.1$$

$$Var[X] = 7.1 - 2.5^2 = 0.85$$

c. $E[Y] = E[(X-2)^2] = 0.2(1-2)^2 + 0.2 * (2-2)^2 + 0.5 * (3-2)^2 + 0.1 * (4-2)^2 = 1$

(G4)

Find the variance of *X*, the random variable with probability mass function

$$p(x) = \begin{cases} (|x-3|+1)/28 & \text{if } x = -3,-2,-1,0,1,2,3 \\ 0 & otherwise. \end{cases}$$

Answer:

$$E(X) = \sum_{x=-3}^{3} xp(x) = -1, E(X^2) = \sum_{x=-3}^{3} x^2 p(x) = 4$$

$V ar(X) = 4 - 1 = 3.$

Hard

(G5)

Question:

Consider four independent rolls of a 6-sided die. Let X be the number of 1's and let Y be the number of 2's obtained. What is the joint PMF of X and Y?

Answer:

$$p_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}, \qquad y = 0, 1, \ldots, 4.$$

To compute the conditional PMF $p_{X|Y}$, note that given that $Y = y$, X is the number of 1's in the remaining $4 - y$ rolls, each of which can take the 5 values $1, 3, 4, 5, 6$ with equal probability $1/5$. Thus, the conditional PMF $p_{X|Y}$ is binomial with parameters $4 - y$ and $p = 1/5$:

$$p_{X|Y}(x \mid y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x},$$

for all nonnegative integers x and y such that $0 \leq x + y \leq 4$. The joint PMF is now given by

$$\begin{aligned} p_{X,Y}(x, y) &= p_Y(y)p_{X|Y}(x \mid y) \\ &= \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}\binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x}, \end{aligned}$$

(G4)

Question:

We roll a standard fair die over and over.
What is the expected number of rolls until the first pair of consecutive sixes appears?
(Hint: The answer is not 36.)

Answer:

- $X :=$ number of rolls until first pair of consecutive $6'$s appears
- $R_i :=$ outcome of i^{th} roll
- $Y =$ number of rolls until first pair of consecutive $6'$s appears (counting from 2^{nd} roll)

$$\begin{aligned} &E[X] \\ &= E[X \mid R_1 \neq 6] \cdot P(R_1 \neq 6) + E[X \mid R_1 = 6] \cdot P(R_1 = 6) \\ &= E[1 + Y \mid R_1 \neq 6] \cdot \frac{5}{6} + E[X \mid R_1 = 6] \cdot \frac{1}{6} \\ &= (E[1 \mid R_1 \neq 6] + E[Y \mid R_1 \neq 6]) \cdot \frac{5}{6} + E[X \mid R_1 = 6] \cdot \frac{1}{6} \\ &= (1 + E[X]) \cdot \frac{5}{6} + E[X \mid R_1 = 6] \cdot \frac{1}{6} \end{aligned}$$

- Z = number of rolls until first pair of consecutive 6's appears (counting from 3^{rd} roll)

$$\begin{aligned}
&E[X \mid R_1 = 6] \\
&= E[X \mid R_1 = 6 \cap R_2 \neq 6] \cdot P(R_2 \neq 6) + E[X \mid R_1 = 6 \cap R_2 = 6] \cdot P(R_2 = 6) \\
&= E[2 + Z \mid R_1 = 6 \cap R_2 \neq 6] \cdot \frac{5}{6} + E[2 + Z \mid R_1 = 6 \cap R_2 = 6] \cdot \frac{1}{6} \\
&= (E[2 \mid R_1 = 6 \cap R_2 \neq 6] + E[Z \mid R_1 = 6 \cap R_2 \neq 6]) \cdot \frac{5}{6} \\
&\quad + (E[2 \mid R_1 = 6 \cap R_2 = 6] + E[Z \mid R_1 = 6 \cap R_2 = 6]) \cdot \frac{1}{6} \\
&= (2 + E[X]) \cdot \frac{5}{6} + (2 + 0) \cdot \frac{1}{6} \\
&= 2 + E[X] \cdot \frac{5}{6}
\end{aligned}$$

- $E[X] = (1 + E[X]) \cdot \frac{5}{6} + E[X \mid R_1 = 6] \cdot \frac{1}{6}$
- $E[X \mid R_1 = 6] = 2 + E[X] \cdot \frac{5}{6}$

$$E[X] = (1 + E[X]) \cdot \frac{5}{6} + \left(2 + E[X] \cdot \frac{5}{6}\right) \cdot \frac{1}{6} = \frac{7}{6} + E[X] \cdot \frac{35}{36}$$

$$\Rightarrow E[X] = 42$$

(G2)

Question:

In the online game, you can buy a key to unlock a treasure box. You will then receive a treasure from the set {coin, sword, shield, potion, gun, grenade}, with equal probability, independent of other treasures. How many keys do you expect to buy before you get each possible treasure at least once?

Answer:

Define ''Success'' as receiving a treasure that has not been received before.

X_i : number of keys spent between ith success and (i+1)th success

First key definitely get a distinctive treasure. So, $X = 1 + \sum_{i=1}^{5} X_i$. Then,

$$E[X] = 1 + \sum_{i=1}^{5} E[X|i].$$

If receiving i different treasures so far, each key will give you another different treasure with probability (6-i)/6. Hence, we know X_i is geometric random variable with probability p = (6-i)/6

$$E[X_i] = 1 / p = \frac{6}{(6-i)}$$

$$E[X] = 1 + \sum_{i=1}^{5} \frac{6}{(6-i)} = 1 + 6\sum_{i=1}^{5} \frac{1}{i} = 14.698$$

(G2)

Question:

The demand for a certain weekly magazine at a newsstand is a random variable with probability mass function p(i) = (10−i)/18, i = 4, 5, 6, 7. If the magazine sells for \$a and costs \$2a/3 to the owner, and the unsold magazines cannot be returned, how many magazines should be ordered every week to maximize the profit in the long run?

Answer:

For $i = 4, 5, 6, 7$, let X_i be the profit if i magazines are ordered. Then

$$E(X_4) = \frac{4a}{3},$$

$$E(X_5) = \frac{2a}{3} \cdot \frac{6}{18} + \frac{5a}{3} \cdot \frac{12}{18} = \frac{4a}{3},$$

$$E(X_6) = 0 \cdot \frac{6}{18} + a \cdot \frac{5}{18} + \frac{6a}{3} \cdot \frac{7}{18} = \frac{19a}{18},$$

$$E(X_7) = -\frac{2a}{3} \cdot \frac{6}{18} + \frac{a}{3} \cdot \frac{5}{18} + \frac{4a}{3} \cdot \frac{4}{18} + \frac{7a}{3} \cdot \frac{3}{18} = \frac{10a}{18}.$$

Since $4a/3 > 19a/18$ and $4a/3 > 10a/18$, either 4, or 5 magazines should be ordered to maximize the profit in the long run.

(G3)

Question:

A fair die is tossed 6 times. What is the expected number of times T that two consecutive pionts occur? For example, if the outcome is 111223 then T = 3.

Answer:

Let Ti be the random variable that takes value 1 if tosses i, i + 1 are the same, and 0 if not. Then T = T1 + T2 +⋯+ T5. By the linearity of expectation E[T] = E[T1] +⋯+ E[T5]. Each Ti takes value 1 with probability 6/36 = 1/6. Therefore E[T] = 5*(1/6) = 5/6

(G2)

Question:

random variable X with pmf $p(X=i)=k*(5t)^i/i!$

Calculate p(X=3)

Answer:

1. Sol k: $\sum p(X=i)=k((5t)+(5t)2/2+(5t)3/3!.....)=k(e^{5t})$

so $k=e^{-5t}$

2. $p(X=3)=e^{-5t}*125t^3/6$

.

TA

(a)

$$\frac{1}{3}+\frac{1}{2}+\frac{1}{18}=\frac{8}{9}$$

(b)

$$1x\frac{1}{3}+2x\frac{1}{2}+3x\frac{1}{18}+4x\frac{1}{9}=\frac{35}{18}$$