Easy

1.

(G5)

At an international class, the student who could speak English account for 70%, one could speak Japanese account for 50%. What is the range of probability of student who can speak both?

(sol)

Let the event student could speak English be called event A, one could speak Japanese be called event B.

Minimum:

In order to make the intersection become smallest, we can assume that student who could speak English or Japanese account for all class. That is, $P(A \cup B) = 1$, and we could know that the intersection would be:

$$P(A) + P(B) - P(A \cap B) = P(A \cup B) = 1$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 70\% + 50\% - 100\% = 20\%$$

Consequently, the minimum probability is 20%

Maximum:

The maximum probability occurs when the intersection is the event itself, that is, one event is the subset of the other one. Of student who could speak English, there is 50% student who could speak both. As a result, the maximum probability of being able to speaking both is 50%.

2.

(G5)

In a graduation travel vote , there are 25 students (80% for boys)vote to Vietnam,10 students(50% for boys) vote to Japan ,and 15 students (5 boys)vote to Korea .Supposed that there are 50 students in this class. If a girl is selected at random, what's the probability that she doesn't vote to Vietnam ?

(sol)

Let A is girl , B is boy , V is voting to Vietnam

Then $P(A) = 2/5$ $P(B) = 3/5$

$P(V^C \mid A)=(P(V^C \cap A))/(P(A)) = 0.75$

3.

(G1)

The coefficients of the quadratic equation $x2 + bx + c = 0$ are determined by tossing a fair die twice (the first outcome is b, the second one is c). Find the probability that the equation has real roots.

Answer :

The quadratic equation has real roots if $b^2 - 4ac \geq 0$. There are several situation to get real roots:

(a) b = 2 and c = 1,

(b) b = 3 and c = 1 or 2,

(c) b = 4 and c = 1 or 2 or 3 or 4

(d) b = 5 and c = 1 or 2 or 3 or 4 or 5 or 6

(e) b = 6 and c = 1 or 2 or 3 or 4 or 5 or 6

The probability P(A) to get real roots is :

$$\frac{1+2+4+6+6}{36} = \frac{19}{36}$$

4.

(G3)

A child's DNA is (A, a). There are two candidates of her father, F_1(a, a) and F_2(A, a), F_1's probability of being the child's father is p, and the other one is 1-p. Now knowing that the child's mom's DNA is (A, A), find the probability that F1 is the child's father.

(sol)

Mi denotes the event Fi is the child's dad.

N denotes the event of the child's DNA is (A, a)

$$P(M_1|N) = \frac{P(N \vee M_1)P(M_1)}{P(N \vee M_1)P(M_1) + P(N \vee M_2)P(M_2)}$$

$$= \frac{1*p}{1*p + \left(\frac{1}{2}\right)(1-p)}$$

$$= \frac{2p}{1+p}$$

5.

(G3)

For three events A, B, and C, we know that A and C are independent,

B and C are independent , A and B are

disjoint ,P(AUC)=2/3,P(BUC)=3/4,P(AUBUC)=11/12 Find P(A),P(B), and P(C).

(sol)

We can use the Venn diagram in Figure 1.26 to better visualize the events in this problem. We assume P(A)=a,P(B)=b, and P(C)=c. Note that the assumptions about independence and disjointness of sets are already included in the figure.

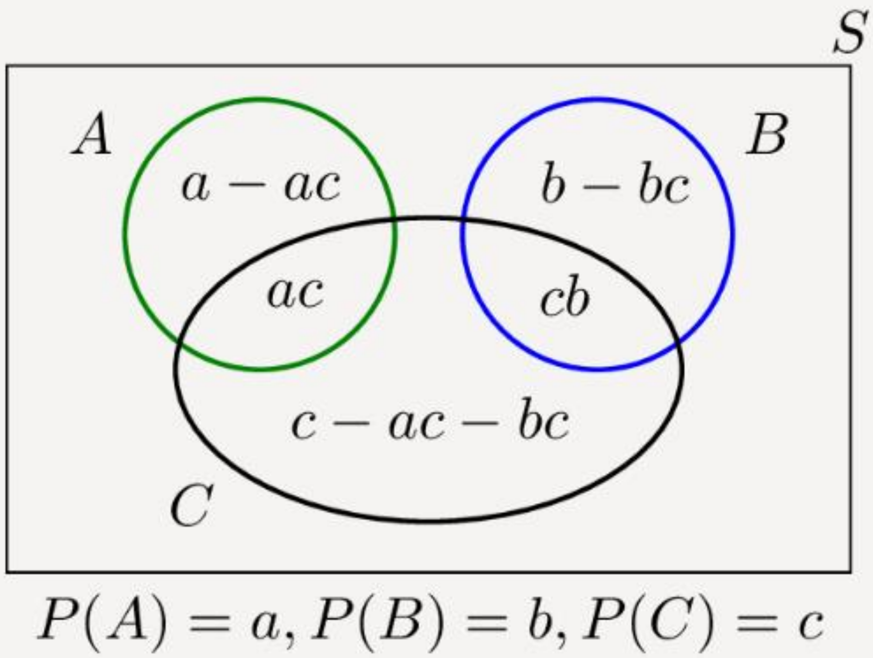
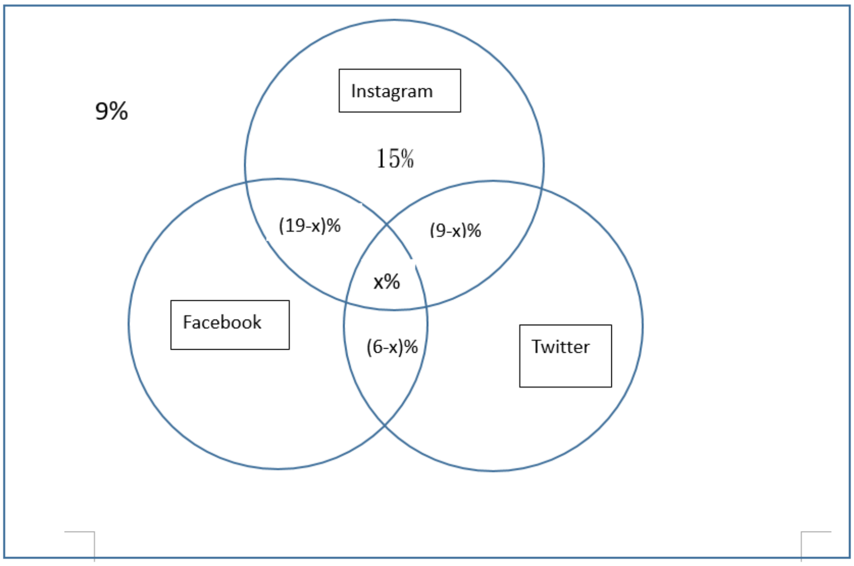


Fig.1.26 - Venn diagram for Problem 3.

Now we can write
$P(A \cup C) = a + c - ac = 2/3$;
$P(B \cup C) = b + c - bc = 3/4$;
$P(A \cup B \cup C) = a + b + c - ac - bc = 11/12$.
By subtracting the third equation from the sum of the first and second equations, we immediately obtain $c = 1/2$, which then gives $a = 1/3$ and $b = 1/2$.

6.
(G2)
In a classroom, 51% of the students have Facebook accounts, 31% have Twitter accounts, 39% have Instagram accounts. 19% have both Instagram and Fackbook accounts, 9% have both Twitter and Instagram accounts, 6% have both Twitter and Facebook accounts, 15% only have Instagram accounts. 9% don't have any accounts. And a student in the classroom is selected randomly, find the probability that he/she only has Twitter account.

(sol)
By the problem, we can draw grpah below,



Set x represent the percentage of students who have all accounts at the same time.
In Instagram circle,

15% + (19-x)% + x% +(9-x)% =39%

⇨ x = 4

In Twitter circle,

31% -(4% + 5% +2%) =20%(answer)

7.

(G6)

In a chest of drawers with n drawers, there is a 1000 NT banknotewith probability p. We open n-1 drawers but we don't find the banknote. What's the probability to find it in the last drawers?

(sol)

Let Ai denote the event "The banknote is in the drawer number i".
Let B denote the event "The banknote is not in the chest of drawers.
Events A1,...,An and B are a complete system of events.
P(A1)=P(A2)=...=P(An)=p/n and P(B)=1-p
We can use conditional probability:

$$P\left(A_n \middle| \overline{A_1} \cap \ldots \cap \overline{A_{n-1}}\right) = \frac{P(\overline{A_1} \cap \ldots \cap \overline{A_{n-1}} \cap An)}{P(\overline{A_1} \cap \ldots \cap \overline{A_{n-1}})}$$

But if the banknote is in the last draw it means that there is no banknote in all the others draws so we have: $P(\overline{A_1} \cap \ldots \cap \overline{A_{n-1}} \cap An) = \frac{p}{n}$

Moreover,

$$P(\overline{A_1} \cap \ldots \cap \overline{A_{n-1}}) = P\left(\overline{A_1 \cup \ldots \cup A_{n-1}}\right) = 1 - P(A_1 \cup \ldots \cup A_{n-1}) = 1 - \frac{(n-1)p}{n}$$

Hard

(G5)

1.A and B are gambling,they flipped a fair coin. If coin lands heads up A takes 1 dollar from B,similarly,if coin lands tails up B takes 1 dollar from A. A has i dollars and B has r dollars at the beginning. What is the probability that A loses all of his money?

(sol)

甲贏乙 = 乙贏甲 = $\frac{1}{2}$

現在甲有10元，乙有1元，所以假設他起點 P_1

輸一元向左走一格，贏一元，向右走一格

如果他從P_0開始走，那就沒機會了 ，因為他已經沒錢了

如果他從P_{11}開始走，那也不用走， 因為他已經贏了

$P_0 = 0, P_{11} = 1$

且這些點皆滿足 $P_n = \frac{1}{2}P_{n+1} + \frac{1}{2}P_{n-1}$

$\frac{1}{2}(P_{n+1} - P_n) = \frac{1}{2}(P_n - P_{n-1})$ $(n \geq 1)$

$P_{n+1} - P_n = P_n - P_{n-1}$ (所以這是一個等差數列)

$p_{11} = 1, p_0 = 0,$ 共差$d = \frac{1}{11}$ $p_n = p_0 + \frac{n}{11} = \frac{n}{11}$

$P_1 = \frac{1}{11}$

2.

(G3)

On the two-dimension coordinate, Sam and Daniel throw the dice twice respectively. The first throw represents the position on x-axis and second represents y-axis, what is the probability that Sam and Daniel form a straight line with the origin(Three Points Collinear)? (Supposed that Sam and Daniel are at different position)

(sol)

To form the straight line, Sam and Daniel should have the same slope.

a. m=1 → (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)
b. m=2 → (1,2), (2,4), (3,6)
c. m=1/2 →(2,1), (4,2), (6,3)
d. m=3 →(1,3), (2,6)

e. m=1/3 →(3,1), (6,2)
f. m=2/3 →(3,2),(6,4)
g. m=3/2 →(2,3),(4,6)

Above are the positions that may form a straight line.

$$P = \frac{\binom{6}{2}*2+\binom{3}{2}*2*2+\binom{2}{2}*2!*2+\binom{2}{2}*2!*2}{6*6*6*6} = \frac{25}{648}$$

3.
(G3)
Suppose there are six dice in a box, one of which is called the "all-6-die", having 6 on all sides, another is called the "all-1-die", having 1 on all sides, the other four dice are all regular, with each face having distinct numbers from 1 to 6. A person reaches into the box and picks two dice and tosses it, the sum of the 2 numbers facing up is 7, what is the probability for the 2 dice being the "all-6-die" and "all-1-die"？

(sol)
Our problem involves 3 cases,
Case 1: numbers 1 and 6 facing up
Case 2: numbers 2 and 5 facing up
Case 3: numbers 3 and 4 facing up

For Case 1, the person has a $\frac{1}{15}$ chance of choosing the "all-6-die" and "all-1-die". In this case, the chances of seeing numbers 1 and 6 facing up is 1. The person also has a $\frac{2}{5}$ chance of choosing two regular dice. In this case, the chances of seeing numbers 1 and 6 facing up is $\frac{1}{18}$. The person also has a $\frac{4}{15}$ chance of choosing a regular die and the "all-6-die". In this case, the chances of seeing numbers 1 and 6 facing up is $\frac{1}{6}$. The person also has a $\frac{4}{15}$ chance of choosing a regular die and the "all-1-die". In this case, the chances of seeing numbers 1 and 6 facing up is $\frac{1}{6}$. Therefore, the probability for Case 1 is $\frac{8}{45}$.
For Case 2, the person has a $\frac{2}{5}$ chance of choosing two regular dice. In this case, the chances of seeing numbers 2 and 5 facing up is $\frac{1}{18}$. Therefore, the probability for Case 2 is $\frac{1}{45}$.
Case 3 has the same probability as Case 2.
To sum up, the probability for the sum of the two numbers being 7 is $\frac{8}{45}+\frac{1}{45}+\frac{1}{45}=\frac{2}{9}$. The probability for the sum of the two numbers being 7 and the two dice being the "all-6-die" and "all-1-die" is $\frac{1}{15}$. Therefore, our answer is $\frac{3}{10}$.

Case 1:

$$\left(\frac{C_2^4}{C_2^6}\times\frac{1}{6}\times\frac{1}{6}\times 2\right)+\left(\frac{C_2^2}{C_2^6}\times 1\right)+\left(\frac{C_1^1\times C_1^4}{C_2^6}\times 1\times\frac{1}{6}\right)+\left(\frac{C_1^1\times C_1^4}{C_2^6}\times 1\times\frac{1}{6}\right)=\frac{8}{45}$$

2 regular dice

"all-6-die" and "all-1-die"

"all-6-die" and regular die

"all-1-die" and regular die

Case 2:

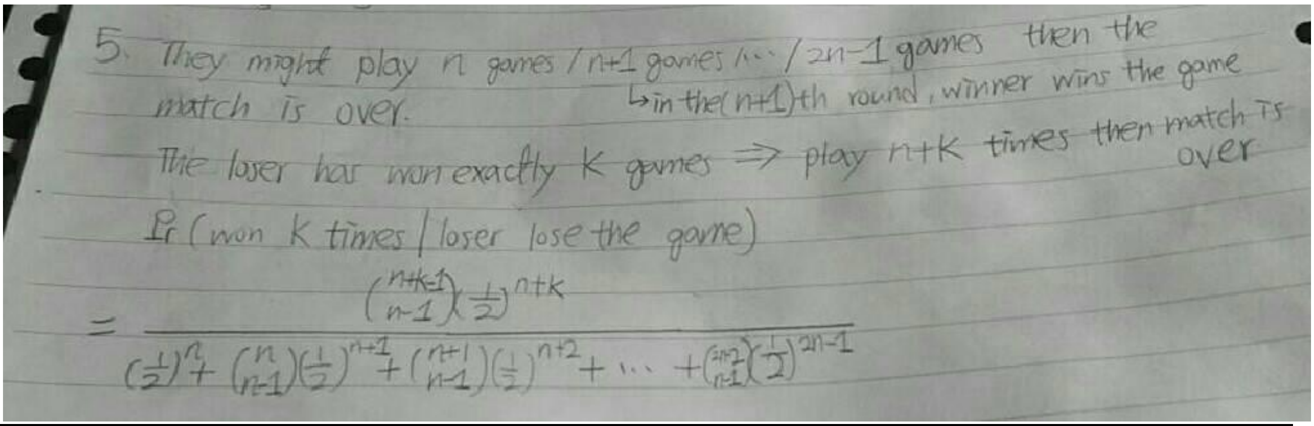
$$\frac{C_2^4}{C_2^6} \times \frac{1}{6} \times \frac{1}{6} \times 2 = \frac{1}{45}$$

Case 3:

$$\frac{C_2^4}{C_2^6} \times \frac{1}{6} \times \frac{1}{6} \times 2 = \frac{1}{45}$$

$$\frac{\frac{1}{15}}{\frac{1}{45} + \frac{1}{45} + \frac{8}{45}} = \frac{3}{10}$$

4.
(G1)
Two persons are playing a match in which they stop as soon as anyone of them wins n games. Suppose that the probability for each person to win any game is 1/2, independent of other games. What is the probability that the loser has won exactly k games when the match is over?



5. They might play n games / n+1 games /... / 2n−1 games then the match is over.
↳ in the (n+1)th round, winner wins the game
The loser has won exactly k games ⇒ play n+k times then match is over
Pr(won k times | loser lose the game)

$$= \frac{\binom{n+k-1}{n-1}\left(\frac{1}{2}\right)^{n+k}}{\left(\frac{1}{2}\right)^n + \binom{n}{n-1}\left(\frac{1}{2}\right)^{n+1} + \binom{n+1}{n-1}\left(\frac{1}{2}\right)^{n+2} + \cdots + \binom{2n-2}{n-1}\left(\frac{1}{2}\right)^{2n-1}}$$

5.
(G1)

Rolling three fair dice today. What is the probability of rolling the smallest dice point is 2 given that the biggest dice point is 5?

Answer 2:

Subset A represents the numbers of event which the biggest point is five is equal to $n(A) = 5^3 - 4^3 = 61$

The numbers of event $n(A \cap B)$ which is correspondent to the requirement of the problem are

(5, 2, 2) = 3 events

(5, 2, 3) = 6 events

(5, 2, 4) = 6 events

(5, 2, 5) = 3 events

So, the answer goes to $P(B|A) = \frac{N(A \cap B)}{n(A)} = \frac{18}{61}$

TA-problem/answer:

(PROBLEM)

Suppose that three numbers are selected one by one, at random and without replacement from the set of numbers {1,2,3,….,n}. What is the probability that the third number falls between the first two if the first number is smaller than the second?

(sol)

Let three numbers be X,Y,Z respectively.

$P(K) = P(1 \le X < Y < Z \le n) = n(E) / n(S)$

$n(S) = P^n{}_3 = n*(n-1)*(n-2)$

$n(E) = n(1 \le X < Y < Z \le n)$

$= \sum^n{}_{y=3} \sum^{y-1}{}_{z=2} \sum^{z-1}{}_{x=1}$

$= \sum^n{}_{y=3} \sum^{y-1}{}_{z=2}(z-1)$

$= \sum^n{}_{y=3} [1+2+\ldots+(y-2)]$

$= \sum^n{}_{y=3} ½ * (y-1) * (y-2)$

$= ½ * \{1*2 + 2*3 + 3*4 + \ldots. + (n-2)(n-1)\}$

$= ½ * \sum^{n-1}{}_{k=1} k(k-1)$

$= n/6 * (n-1)(n-2)$

$P(1 \le X < Y < Z \le n) = n(E) / n(S) = 1/6$

$P(K') = P(1 \le X < Z \le n) = n(E') / n(S')$

$n(S') = P^n{}_2 = n*(n-1)$

$n(E') = \sum^n{}_{z=2} \sum^{z-1}{}_{x=1}$

$= (1+2+..+n-1)$

$= ½ * n(n-1)$

$P(1 \le X < Z \le n) = ½$

$\therefore P(K|K') = P(K \cap K') / P(K') = P(K) / P(K') = 1/3$