Final Exam Solution

Problem 1 (20 = 10 + 10)

You enter a chess tournament where your probability of winning a game is 0.3 against half of the player (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). Now you play against a randomly chosen opponent

(a) What is the probability of winning?

(b) Suppose that you win, what is the probability that you had an opponent of type 1?

Solution:

a)

Let Ai be the event of playing against opponent of type i

P(A1) = 0.5, P(A2) = 0.25, P(A3) = 0.25

Now, let B be the event of winning, we have

P(B|A1) = 0.3, P(B|A2)=0.4, P(B|A3) = 0.5

Now, remember the total probability theorem

P(B) = P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3)

= 0.375

b)

Let's write out the details again,

$P(A_1) = 0.5$, $P(A_2) = 0.25$, $P(A_3) = 0.25$

$P(B|A_1) = 0.3$, $P(B|A_2)=0.4$, $P(B|A_3) = 0.5$

What needs to be computed? $P(A_1|B)$

Let's recall Bayes rule:

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}$$
$$= \frac{0.5 * 0.3}{0.5 * 0.3 + 0.25 * 0.4 + 0.25 * 0.5} = 0.4$$

Problem 2 (20 = 10 + 10)

A man has a special hen which lays jewelry instead of laying eggs, and her name is "Miracken" (奇雞).

Set the weight of jewelry (g) that Mira lays in this month is a random variable X, and its probability mass function is:

$$P(X = 10) = 0.1,\ P(X = 15) = 0.3,\ P(X = 20) = 0.3,\ P(X = 25) = 0.3$$

While the random variable Y represents the price of jewelry ($USD/g) at the end of this month, and the probability mass function is:

$$P(Y = 3.5) = 0.4,\ P(Y = 4) = 0.2,\ P(Y = 4.5) = 0.4.$$

It is assumed that X and Y are independent.

(a) If he sells out all of the jewelry at the end of this month, what is the expected amount much money will he get?

(b) Continue from (a), what's the variance of the money he gets?

Solution:

a)

E(XY)=E(X)E(Y)=(10*0.1+15*0.3+20*0.3+25*0.3)(3.5*0.4+4.0*0.2+4.5*0.4)=19*4=$76 (USD)

b)

Var[XY]=E[(XY)^2]-(E[XY])^2=E[X^2]E[Y^2]-(E[X]E[Y])^2={Var[X]+(E[X])^2}{Var[Y]+(E[Y])^2}-{(E[X]E[Y])^2}

Var[X]=(-9)^2*0.1+(-4)^2*0.3+1*0.3+6^2*0.3=24

Var[Y]=(-0.5)^2*0.4+0.5^2*0.4=0.2

→Var[XY]=(24+361)(0.2+16)-5776=461

Problem 3 (10)

A floor has parallel lines on it at equal distances l from each other. A needle of length $a < l$ is dropped at random onto the floor. Find the probability that the needle will intersect a line. (This problem is known as Buffon's needle problem.)

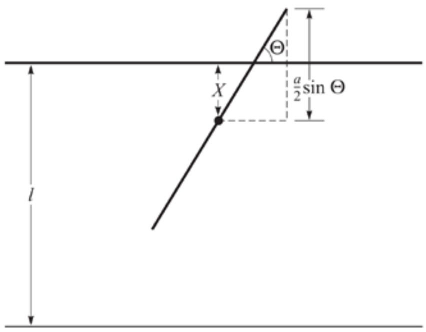

Fig 1

Solution:

Let X be a random variable that gives the distance of the midpoint of the needle to the nearest line (Fig. 1). Let Θ be a random variable that gives the acute angle between the needle (or its extension) and the line. We denote by x and θ any particular values of X and Θ. It is seen that X can take on any value between 0 and $l/2$, so that $0<x<l/2$, Also Θ can take on any value between 0 and $2/\pi$. It follows that

$$P(x < X \le x + dx) = \frac{2}{l}\,dx \qquad P(\theta < \Theta \le \theta + d\theta) = \frac{2}{\pi}\,d\theta$$

i.e., the density functions of X and Θ are given by $f1(x)= \frac{2}{l}$, $f2(\theta) = \frac{\pi}{2}$. As a check, we note that

$$\int_0^{l/2} \frac{2}{l}\,dx = 1, \qquad \int_0^{\pi/2} \frac{2}{\pi}\,d\theta = 1$$

Since X and Θ are independent the joint density function is

$$f(x,\theta) = \frac{2}{l}\cdot\frac{2}{\pi} = \frac{4}{l\pi}$$

From Fig.1 it is seen that the needle actually hits a line when $X \le (a/2)\sin\Theta$. The probability of this event is given by

$$\frac{4}{l\pi}\int_{\theta=0}^{\pi/2}\int_{x=0}^{(a/2)\sin\Theta} dxd\theta = \frac{2a}{l\pi}$$

Problem 4 (15 = 5 + 5 + 5)

Let X be a standard normal random variable, i.e., its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

(a) Find the mean and variance of X^2.

(b) Find the probability density function of X^2.

(c) Let $\{X_1, X_2, ...\}$ be a sequence of independent standard normal random variables.

Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2$. Find

$$\lim_{n\to\infty} P(S_n \le n + 2\sqrt{2n})$$

Solution:

(a)

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int_{-\infty}^{\infty} x^4 e^{-ax^2}dx = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}$$

$$E[X^2] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{2}}dx = \frac{1}{\sqrt{2\pi}} * \frac{1}{2}\sqrt{8\pi} = 1$$

$$E[X^4] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{\frac{-x^2}{2}}dx = \frac{1}{\sqrt{2\pi}} * \frac{3}{4}\sqrt{32\pi} = 3$$

$$\mathrm{Var}[X^2] = E[X^4] - E[X^2]^2 = 3 - 1 = 2$$

(b) Let $Y = X^2$. If $x < 0$, $x = -\sqrt{y}$ and $x > 0$, $x = \sqrt{y}$.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{\frac{(-\sqrt{y})^2}{2}}\left|\frac{d}{dy}\right|(-\sqrt{y}) + \frac{1}{\sqrt{2\pi}}e^{\frac{(\sqrt{y})^2}{2}}\left|\frac{d}{dy}\right|(\sqrt{y})$$

$$= \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi y}}e^{\frac{-y}{2}} \qquad (y > 0)$$

(c) $E[X^2] = \mu = 1$ and $\mathrm{Var}[X^2] = \sigma^2 = 2$. For $S_n \le n + 2\sqrt{2n}$,

$$Z = \frac{S_n - n}{2\sqrt{2n}} = \frac{S_n - \mu}{\sigma\sqrt{n}} \le 2$$

By Central limit theory,

$$\lim_{n\to\infty} P(S_n \le n + 2\sqrt{2n}) = \lim_{n\to\infty} P\left(\frac{S_n - \mu}{\sigma\sqrt{n}} \le 2\right) = \Phi(2) = 0.9772$$

where $\Phi(x)$ is a normal Gaussian CDF.

Problem 5 (15 = 10 + 5)

(1) Heights of 10 year-olds, regardless of gender, closely follow a normal distribution with mean 55 inches and standard deviation 6 inches. Which of the following is true (derivation is needed to get full credit)
   (A) We would expect more number of 10 year-olds to be shorter than 55 inches than the number of them who are taller than 55 inches
   (B) Roughly 95% of 10 year-olds are between 37 and 73 inches tall
   (C) A 10-year-old who is 65 inches tall would be considered more unusual than a 10-year-old who is 45 inches tall
   (D) None of these

(2) A fly has a life between 4-6 days. What is the probability that the fly will die at exactly 5 days? (full derivation is needed to get full credit)
   (A) 1/2
   (B) 1/4
   (C) 1/3
   (D) Nearly 0

Solution:

(1) (D)

(2) (D)

Here since the probabilities are continuous, the probabilities form a mass function. The probability of a certain event is calculated by finding the area under the curve for the given conditions. Here since we're trying to calculate the probability of the fly dying at exactly 5 days – the area under the curve would be 0. Also to come to think of it, the probability if dying at exactly 5 days is impossible for us to even figure out since we cannot measure with infinite precision if it was exactly 5 days.

Problem 6 (20 = 10 + 10)

Sarah is working in a photon lab. She measures the number of photons emitted by a source every one second interval. The number of photons emitted at each interval i is random and denoted by the random variable X_i, without any pre-defined probability distribution function. These observations are assumed to be i.i.d. with a finite mean, λ, and a finite variance, v. During her study, she collects the data and calculates the sample mean estimator,

$$M_n = \frac{1}{n}\sum_{i}^{n} X_i$$

And a variance estimator,

$$\hat{S}_n^2 = \frac{1}{n-1}\sum_{i}^{n}(X_i - M_n)^2$$

(1) Is this variance estimator an unbiased estimator for variance, v? (show your full proof to receive credit)

(2) Her experiment shows that the sample mean $m_n = 7.1$, and the variance estimator, she finds $\hat{s}_n^2 = 7.3$. One day, a huge typhoon comes and sweeps away all of the raw data except these two estimated

values. The lab is in need to make probabilistic inference, her boss asked her to specify (assume) the distribution of X_i as Poisson distribution:

$$p_{X_i}(x) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0,1,2 \ldots$$

and further request her to find the Maximum Likelihood (ML) the variance, v, of X_i. Sarah quickly remembers that in Poisson distribution, the expected value and the variance are the same ($\lambda = v$), however the only two values that she has are the two estimated values ($m_n = 7.1,\ \hat{s}_n^2 = 7.3$) are different.

Does she now have enough information to calculate the ML estimate of the variance, v, of X_i, given that her boss asked her to model this as Poisson distribution? If so, what is it, If not, why not? (full derivation is needed to receive credit)

Solution:

(1)

We want to find $\mathrm{E}(\hat{S}_n^2) = \mathrm{v}$ to show unbiased-ness.

$$\mathrm{E}(\hat{S}_n^2) = \mathrm{E}\left[\frac{1}{n-1}\left(\sum_i^n (X_i - M_n)^2\right)\right] = E\left[\frac{1}{n-1}\left(\sum_i^n X_i^2 - 2X_iM_n + M_n^2\right)\right]$$

$$= \frac{1}{n-1}\left(E\left[\sum_i^n X_i^2\right] - nE[M_n^2]\right)$$

Now try to solve for each term:

$$E\left[\sum_i^n X_i^2\right] = \sum_i^n E[X_i^2] = \mathrm{n}(var(X_i) + E[X_i]^2) = \mathrm{nv} + \mathrm{n}\lambda^2$$

$$nE[M_n^2] = nE\left[\left(\frac{1}{n}\sum_i^n X_i\right)^2\right] = \frac{1}{n}E\left[\left(\sum_i^n X_i\right)^2\right] = \frac{1}{n}\left((n^2-n)\lambda^2 + \sum_i^n E[X_i^2]\right)$$

$$= \frac{1}{n}\left((n^2-n)\lambda^2 + \sum_i^n var(X_i) + E[X_i]^2\right) = nE\left[\lambda^2 + \frac{v}{n}\right]$$

Put it together,

$$\mathrm{E}(\hat{S}_n^2) = \frac{1}{n-1}\left(E\left[\sum_i^n X_i^2\right] - nE[M_n^2]\right) = \frac{1}{n-1}(nv + \mathrm{n}\lambda^2 - n\lambda^2 - v) = v$$

(2)

Yes, she has enough information. Let’s first find ML estimate of the parameter λ for Poisson distribution. The log-likelihood function (assume IID) is:

$$\log P_X(x_1, x_2, x_3, \ldots x_n; \lambda) = \sum_{i-1}^{n} logP_X(x_i; \lambda) = -n\lambda + \sum_{i=1}^{n} x_i log\lambda - \sum_{i=1}^{n} log(x_i!)$$

Find derivative and set it equal 0,

$$0 = -\mathrm{n} + \frac{1}{\lambda}\sum_{i}^{n} x_i$$

The ML estimator for λ is:

$$\hat{\lambda}_n = \frac{1}{n}\sum_{i}^{n} X_i$$

This ML estimator is the same as the sample mean estimator M_n=7.1

Since we know for Poisson distribution the variance parameter equal the expected value parameter equals to λ. The ML estimator for variance of Poisson in this case is the same as the sample mean estimator = 7.1

BONUS (10)

Assume i.i.d observations $X_{1,}X_{2,}X_{3,} \ldots, X_{n,}$ that are uniformly distributed over the interval $[\theta, \theta + 1]$. Find a ML (Maximum Likelihood) estimate for θ ? If there's more than one, find all of them.

Solution:

The PDF is:

$$f_{X_i}(x_i) = \begin{cases} 1, & \text{if } \theta \le x_i \le \theta + 1 \\ & 0, \text{otherwise} \end{cases}$$

The likelihood function (assume IID):

$$f_{X_i}(x_1, x_2, x_3, \ldots x_n; \theta) = f_{X_1}(x_1; \theta) f_{X_n}(x_n; \theta) \ldots f_{X_n}(x_n; \theta)$$

$$= \begin{cases} 1, & \text{if } \theta \le \min_{i=1,\ldots n} x_i \le \max_{i=1,\ldots n} x_i \le \theta + 1 \\ & 0, \text{otherwise} \end{cases}$$

Any value in the interval:

$$\left[\min_{i=1,\ldots n} x_i \le \max_{i=1,\ldots n} x_i\right]$$

Would maximize the likelihood function and is therefore a ML estimator