10720 EECS 303003 Probability Homework #5 Due: 6/5 21:00

1. A statistician wants to estimate the mean height h (in meters) of a population, based on n independent samples $X_1, X_2, \dots, X_n$, chosen uniformly from the entire population. He uses the sample mean $M_n = (X_1, X_2, \dots, X_n)/n$ as the estimate of h, and a rough guess of 1.0 meters for the standard deviation of the samples X_i.
   (a) How large should n be so that the standard deviation of M_n is at most 1 centimeter?
   (b) How large should n be so that Chebyshev's inequality guarantees that the estimate is within 5 centimeters from h, with probability at least 0.99?

2. A twice differentiable real-valued function f or a convex if its second derivative $(d^2f/dx^2)(x)$ is nonnegative for all x in its domain of definition.
   (a) Show that if f is twice differentiable and convex, then the first order Taylor f is an underestimate of the function. that is,
   $$f(a) + (x - a)\frac{df}{dx}(a) \leq f(x)$$
   for every a and x.
   (b) Show that if f has the property in part (b), and if X is a random variable, then
   $$f(E[X]) \leq E[f(X)]$$

3. Suppose the grades in a finite mathematics class are Normally distributed with a mean of 75 and a standard deviation of 5.
   (a) What's the probability that a randomly selected student had a grade of at least 83?
   (b) What is the probability that the average grade for 5 randomly selected students was at least 83?

4. Let $X_1, Y_1, X_2, Y_2 \dots\dots$ be independent random variables, uniformly distributed in the unit interval [0,1], and let
   $$\mathrm{W} = \frac{(X_1 + \cdots + X_{16}) - (Y_1 + \cdots + Y_{16})}{16}$$
   Find a numerical approximation to the quantity
   $$\mathrm{P}\,(|\mathrm{W} - \mathrm{E[W]}| < 0.001)$$

5. Let $X_1, X_2, \dots\dots$ be a sequence of independent identically distributed zero-mean random variables with common variance σ^2, and associated transform $M_x(s)$. We

assume that $M_x(s)$ is finite when $-\mathrm{d} < \mathrm{s} < \mathrm{d}$, where d is some positive number. Let

$$Z_n = \frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}}$$

(a) Show that the transform associated with Z_n satisfies

$$M_{Z_n}(\mathrm{s}) = \left(M_x\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n$$

(b) Suppose that the transform $M_x(s)$ has a second order Taylor series expansion around s = 0, of the form

$$M_x(s) = a + bs + cs^2 + o(s^2)$$

Where $o(s^2)$ is a function that satisfies $\lim_{\mathrm{s}\to 0} \frac{o(s^2)}{s^2} = 0$. Find a, b, and c in terms of σ^2 .

(c) Combine the results of parts (a) and (b) to show that the transform $M_{Z_n}(\mathrm{s})$ converges to the transform associated with a standard normal random variable, that is,

$$\lim_{\mathrm{n}\to\infty} M_{Z_n}(\mathrm{s}) = e^{\frac{s^2}{2}} \quad \text{, for all s.}$$

Note: The central limit theorem follows from the result of part (c), together with the fact (whose proof lies beyond the scope of the text) that if the transforms $M_{Zn}(\mathrm{s})$ converge to the transform $M_Z(\mathrm{s})$ of a random variable Z whose CDF is continuous, then the CDFs F_{Zn} converge to the CDF of Z. In our case, this implies that the CDF of Z_n converges to the CDF of a standard normal.

Z-tables : Probability of a larger value

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.10	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.20	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.30	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.40	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.50	**0.3085**	**0.3050**	**0.3015**	**0.2981**	**0.2946**	**0.2912**	**0.2877**	**0.2843**	**0.2810**	**0.2776**
0.60	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.70	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.80	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.90	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.00	**0.1587**	**0.1562**	**0.1539**	**0.1515**	**0.1492**	**0.1469**	**0.1446**	**0.1423**	**0.1401**	**0.1379**
1.10	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.20	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.30	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.40	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.50	**0.0668**	**0.0655**	**0.0643**	**0.0630**	**0.0618**	**0.0606**	**0.0594**	**0.0582**	**0.0571**	**0.0559**
1.60	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.70	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.80	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.90	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.00	**0.0228**	**0.0222**	**0.0217**	**0.0212**	**0.0207**	**0.0202**	**0.0197**	**0.0192**	**0.0188**	**0.0183**
2.10	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.20	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.30	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.40	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.50	**0.0062**	**0.0060**	**0.0059**	**0.0057**	**0.0055**	**0.0054**	**0.0052**	**0.0051**	**0.0049**	**0.0048**
2.60	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.70	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.80	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.90	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.00	**0.0013**	**0.0013**	**0.0013**	**0.0012**	**0.0012**	**0.0011**	**0.0011**	**0.0011**	**0.0010**	**0.0010**
3.10	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.20	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.30	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.40	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
3.50	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**	**0.0002**
3.60	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.70	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.80	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**	**0.0000**