10720 EECS 303003 Probability Homework #5 SOLUTION

1. **A statistician wants to estimate the mean height h (in meters) of a population, based on n independent samples $X_1, X_2, \dots, X_n$, chosen uniformly from the entire population. He uses the sample mean $M_n = (X_1, X_2, \dots, X_n)/n$ as the estimate of h, and a rough guess of 1.0 meters for the standard deviation of the samples X_i.**
   - **(a) How large should n be so that the standard deviation of M_n is at most 1 centimeter?**
   - **(b) How large should n be so that Chebyshev's inequality guarantees that the estimate is within 5 centimeters from h, with probability at least 0.99?**

ANS:

(a) We have $\sigma_{M_n} = \frac{1}{\sqrt{n}}$, so in order that $\sigma_{M_n} \leq 0.01$, we must have $n \geq 10{,}000$.

(b) We want to have

$$P(|M_n - h| \leq 0.05) \geq 0.99$$

Using the facts that $\mathrm{h} = \mathrm{E}[M_n], \sigma_{M_n}^2 = \frac{1}{n}$, and the Chebyshev inequality, we have

$$P(|M_n - h| \leq 0.05) = P(|M_n - \mathrm{E}[M_n]| \leq 0.05)$$

$$= 1 - P(|M_n - \mathrm{E}[M_n]| \geq 0.05) \geq 1 - \frac{\frac{1}{n}}{(0.05)^2}$$

Thus, we must have

$$1 - \frac{\frac{1}{n}}{(0.05)^2} \geq 0.99$$

Which yield $n \geq 40{,}000$.

2. **A twice differentiable real-valued function f or a convex if its second derivative $(d^2f/dx^2)(x)$ is nonnegative for all x in its domain of definition.**
   - **(a) Show that if f is twice differentiable and convex, then the first order Taylor f is an underestimate of the function. that is,**

$$f(a) + (x - a)\frac{df}{dx}(a) \leq f(x)$$

for every a and x.

   - **(b) Show that if f has the property in part (b), and if X is a random variable, then**

$$\boldsymbol{f(E[X]) \leq E[f(X)]}$$

ANS:

(a) Since the second derivative of f is nonnegative, its first derivative must be nondecreasing. Using the fundamental theorem of calculus, we obtain

$$\mathrm{f(x)} = f(a) + \int_a^x \frac{df}{dt}(t)dt \;\geq\; f(a) + \int_a^x \frac{df}{dt}(a)dt = f(a) + (x-a)\frac{df}{dx}(a)$$

(b) Since the inequality from part (a) is assumed valid for every possible value x of the random variable X, we obtain

$$f(a) + (X-a)\frac{df}{dx}(a) \leq f(X)$$

We now choose a = E[X] and take expectation, to obtain

$$f(\mathrm{E[X]}) + (\mathrm{E[X]} - \mathrm{E[X]})\frac{df}{dx}(\mathrm{E[X]}) \;\leq\; \mathrm{E}[f(\mathrm{X})],$$

$$\text{or } f(\mathrm{E[X]}) \leq \mathrm{E}[f(\mathrm{X})]$$

3. Suppose the grades in a finite mathematics class are Normally distributed with a mean of 75 and a standard deviation of 5.

(a) What's the probability that a randomly selected student had a grade of at least 83?

(b) What is the probability that the average grade for 5 randomly selected students was at least 83?

ANS:

(a) Since X ~ N(75,5), we can write down that

$$\mathrm{z} = \frac{83-75}{5} = 1.6$$

Therefore

$$\mathrm{P(X \geq 83)} = P(z \geq 1.6) = 1 - 0.9452 = 0.0548$$

(b) Since $\hat{X}$ ~ N(75,$\sqrt{5}$), we can write down that

$$\mathrm{z} = \frac{83-75}{\sqrt{5}} = 3.57$$

Therefore

$$\mathrm{P(X \geq 83)} = P(z \geq 3.57) = 1 - 0.9999 = 0.0001$$

4. Let $\boldsymbol{X_1, Y_1, X_2, Y_2}$ …… be independent random variables, uniformly distributed in the unit interval [0,1], and let

$$\mathbf{W} = \frac{(\boldsymbol{X_1} + \cdots + \boldsymbol{X_{16}}) - (\boldsymbol{Y_1} + \cdots + \boldsymbol{Y_{16}})}{\mathbf{16}}$$

Find a numerical approximation to the quantity

$$\mathbf{P}\left(|\mathbf{W}-\mathbf{E}[\mathbf{W}]|<\mathbf{0.001}\right)$$

ANS:

Note that W is a sample mean of 16 independent identically distributed random variables of the form $X_i - Y_i$, and a normal approximation is appropriate. The random variable $X_i - Y_i$ have zero mean, and variance equal to $\frac{2}{12}$. Therefore, the mean of W is zero, and its variance is $\frac{\frac{2}{12}}{16} = \frac{1}{96}$. Thus,

$$P(|W| \leq 0.001) = P\left(\frac{|W|}{\sqrt{\frac{1}{96}}} < \frac{0.001}{\sqrt{\frac{1}{96}}}\right) \approx \Phi\left(0.001\sqrt{96}\right) - \Phi\left(-0.001\sqrt{96}\right)$$

$$= 2\Phi\left(0.001\sqrt{96}\right) - 1 = 2\Phi(0.0098) - 1 \approx 2 * 0.504 - 1$$
$$= 0.008$$

Let us also point out a somewhat different approach that bypass the need for the normal table. Let Z be a normal ransom variable with zero mean and standard deviation equal to $\frac{1}{\sqrt{96}}$. The standard deviation of Z, which is about 0.1, is much larger than 0.001. Thus, within the interval [-0.001, 0.001], the PDF of Z is approximately constant. Using the formula $P(z-\delta \leq Z \leq z+\delta) \approx f_Z(z) * 2\delta$, with $z = 0$ and $\delta = 0.001$, we obtain

$$P(|W| \leq 0.001) \approx P(-0.001 \leq Z \leq 0.001) \approx f_Z(0) * 0.002 = \frac{0.002}{\sqrt{2\pi}\left(\frac{1}{\sqrt{96}}\right)}$$

$$= 0.0078$$

5. Let $X_1, X_2, \ldots\ldots$ be a sequence of independent identically distributed zero-mean random variables with common variance σ^2, and associated transform $M_x(s)$. We assume that $M_x(s)$ is finite when $-d < s < d$, where d is some positive number. Let

$$Z_n = \frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}}$$

(a) Show that the transform associated with Z_n satisfies

$$M_{Z_n}(s) = \left(M_x\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n$$

(b) Suppose that the transform $M_x(s)$ has a second order Taylor series

expansion around s = 0, of the form

$$M_x(s) = a + bs + cs^2 + o(s^2)$$

Where $o(s^2)$ is a function that satisfies $\lim_{s\to 0} \frac{o(s^2)}{s^2} = 0$. Find a, b, and c in terms of σ^2 .

(c) Combine the results of parts (a) and (b) to show that the transform $M_{Z_n}(s)$ converges to the transform associated with a standard normal random variable, that is,

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{\frac{s^2}{2}} \quad \text{, for all s.}$$

***Note:* The central limit theorem follows from the result of part (c), together with the fact (whose proof lies beyond the scope of the text) that if the transforms $M_{Zn}(s)$ converge to the transform $M_Z(s)$ of a random variable Z whose CDF is continuous, then the CDFs F_{Zn} converge to the CDF of Z. In our case, this implies that the CDF of Z_n converges to the CDF of a standard normal.**

ANS:

(a) We have, using the independence of the X_i,

$$\begin{aligned} M_{Z_n}(s) &= \mathbf{E}\left[e^{sZ_n}\right] \\ &= \mathbf{E}\left[\exp\left\{\frac{s}{\sigma\sqrt{n}}\sum_{i=1}^{n} X_i\right\}\right] \\ &= \prod_{i=1}^{n} \mathbf{E}\left[e^{sX_i/(\sigma\sqrt{n})}\right] \\ &= \left(M_X\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n. \end{aligned}$$

(b) Using the moment generating properties of the transform, we have

$$a = M_X(0) = 1, \qquad b = \frac{d}{ds} M_X(s)\bigg|_{s=0} = \mathbf{E}[X] = 0,$$

and

$$c = \frac{1}{2} \cdot \frac{d^2}{ds^2} M_X(s)\bigg|_{s=0} = \frac{\mathbf{E}[X^2]}{2} = \frac{\sigma^2}{2}.$$

(c) We combine the results of parts (a) and (b). We have

$$M_{Z_n}(s) = \left(M_X\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n = \left(a + \frac{bs}{\sigma\sqrt{n}} + \frac{cs^2}{\sigma^2 n} + o\left(\frac{s^2}{\sigma^2 n}\right)\right)^n,$$

and using the formulas for a, b, and c from part (b), it follows that

$$M_{Z_n}(s) = \left(1 + \frac{s^2}{2n} + o\left(\frac{s^2}{\sigma^2 n}\right)\right)^n.$$

We now take the limit as $n \to \infty$, and use the identity

$$\lim_{n\to\infty}\left(1 + \frac{c}{n}\right)^n = e^c,$$

to obtain

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2}.$$