10720 EECS 303003 Probability Homework #4 Solution

1. Given the joint density function of X, Y as following

$$\mathbf{f}_{XY}(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{2x+y}{a}, & 0 \le x \le 2, \quad 0 \le y \le 4 \\ 0 \quad , & \text{otherwise} \end{cases}$$

(a) Find the value of a.

$$\int_0^4 \int_0^2 \frac{2x+y}{a} dxdy = \frac{1}{a}\int_0^4 4+2ydy = \frac{32}{a} = 1 \rightarrow \mathrm{a} = 32$$

(b) Find the covariance of X and Y, $Cov(x,y)$.

$$\mathrm{E[XY]} = \int_0^4 \int_0^2 xy\frac{2x+y}{32} dxdy = \frac{8}{3}$$

$$\mathrm{E[X]} = \int_0^4 \int_0^2 x\frac{2x+y}{32} dxdy = \frac{7}{6}$$

$$\mathrm{E[XY]} = \int_0^4 \int_0^2 y\frac{2x+y}{32} dxdy = \frac{7}{3}$$

(c) Find the correlation coefficient of X and Y, ρ_{XY}.

$$\rho^2{}_X = E[X^2] - E[X]^2$$

$$= \int_0^4 \int_0^2 x^2\frac{2x+y}{32} dxdy - (\frac{7}{6})^2 = \frac{11}{36}$$

$$\rho^2{}_y = E[Y^2] - E[Y]^2$$

$$= \int_0^4 \int_0^2 y^2\frac{2x+y}{32} dxdy - (\frac{7}{3})^2 = \frac{11}{9}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\rho_X\rho_Y} = \frac{\frac{-1}{18}}{\sqrt{\frac{11}{36}}\sqrt{\frac{11}{9}}} = \frac{-1}{11}$$

2. Suppose that n people have their hats returned at random. Let $X_i = 1$ if the ith person gets his/her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the total number of people who get their own hats back. Please calculate the followings:

(a) $E[X_i^2]\rightarrow$

$$\mathrm{P}(X_i = 1) = \frac{(n-1)!}{n!} = \frac{1}{n};\ P(X_i = 0) = 1 - \frac{1}{n}$$

$$E[X_i^2] = 1x\frac{1}{n} = \frac{1}{n}$$

(b) $E[X_i \cdot X_j], i \neq j\rightarrow$

$$\mathrm{P}(X_i = 1, X_j = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-2)}$$

$$E[X_i X_j] = 1x1x\frac{1}{n(n-1)} = \frac{1}{n(n-1)}$$

(c) $E[S_n^2]\rightarrow$

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$E[S_n^2] = E[X_1^2] + \cdots + E[X_n^2] + 2E[X_1X_2] + \cdots + 2E[X_{n-1}X_n]$$

$$= nx\frac{1}{n} + 2\left\{C_2^n x\frac{1}{n(n-1)}\right\} = 1 + 1 = 2$$

(d) $Var[S_n]$

$$Var(S_n) = E[S_n^2] - E[S_n]^2 = 2 - 1 = 1,$$

where $E[S_n] = E[X_n] + \cdots + E[X_n] = nx\frac{1}{n} = 1$

3. Romeo and Juliet have a date at a given time, denote that random variable X and Y is the amount of time where Romeo and Juliet are late respectively. Assume X and Y are independent and exponentially distributed with different parameters λ and μ, respectively. Find the PDF of X – Y.

$$F_Z(z) = P(X - Y \leq z) = 1 - P(X - Y > z) = 1 - \int_0^\infty \left(\int_{z+y}^\infty f_{X,Y}(x,y)dx \right) dy$$

$$= 1 - \int_0^\infty \mu e^{-\mu y} \left(\int_{z+y}^\infty \lambda e^{-\lambda x} dx \right) dy = 1 - \int_0^\infty \mu e^{-\mu y} e^{-\lambda(z+y)} dy$$

$$= 1 - e^{-\lambda z} \int_0^\infty \mu e^{-(\lambda+\mu)} dy = 1 - \frac{\mu}{\lambda+\mu} e^{-\lambda z}$$

For the case $z < 0$, we have using the preceding calculation

$$F_Z(z) = 1 - F_Z(-z) = 1 - \left(1 - \frac{\lambda}{\lambda+\mu} e^{-\mu(-z)}\right) = \frac{\lambda}{\lambda+\mu} e^{\mu z}$$

Combining the two cases $z \geq 0$ and $z < 0$, we obtain

$$F_z(z) = \begin{cases} 1 - \dfrac{\mu}{\lambda + \mu} e^{-\lambda z}, & if\ z \geq 0 \\ \dfrac{\lambda}{\lambda + \mu} e^{\mu z}, & if\ z < 0 \end{cases}$$

The PDF of Z is obtained by differentiating its CDF. We have

$$f_z(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda + \mu} e^{-\lambda z}, & if\ z \geq 0 \\ \dfrac{\lambda\mu}{\lambda + \mu} e^{\mu z}, & if\ z < 0 \end{cases}$$

4. **The random variables X, Y, and Z are independent and uniformly distributed between zero and one. Find the PDF of X + Y + Z.**

Let V = X + Y. the PDF of V is

$$f_v(v) = \begin{cases} v, & 0 \leq v \leq 1 \\ 2 - v, & 1 \leq v \leq 2 \\ 0, & otherwise \end{cases}$$

Let W = X+Y+Z=V+Z. We convolve the PDFs f_v and f_z, to obtain

$$f_w(w) = \int f_v(v) f_z(w - v) dv$$

We first need to determine the limits of the integration. Since $f_v(v) = 0$ outside the range $0 \leq v \leq 2$, and $f_w(w - v)$=0 outside the range $0 \leq w - v \leq 1$, we see that the integrand can be nonzero only if

$$0 \leq v \leq 2, \text{ and } w - 1 \leq v \leq w.$$

We consider three separate cases. If $w \leq 1$, we have

$$f_w(w) = \int_0^w f_v(v) f_z(w - v) dv = \int_0^w \mathrm{vdv} = \frac{w^2}{2}$$

If $1 \leq w \leq 2, we\ have$

$$f_w(w) = \int_{w-1}^{w} f_v(v) f_z(w - v) dv = \int_{w-1}^{1} v dv + \int_1^w (2 - v) dv$$

$$= \frac{1}{2} - \frac{(w-1)^2}{2} - \frac{(w-2)^2}{2} + \frac{1}{2}$$

Finally. If $2 \leq w \leq 3, we\ have$

$$f_w(w) = \int_{w-1}^{2} f_v(v) f_z(w - v) dv = \int_{w-1}^{2} (2 - v) dv = \frac{(3 - w)^2}{2}$$

To summarize,

$$f_w(w) = \begin{cases} \frac{w^2}{2}, & 0 \le w \le 1 \\ 1 - \frac{(w-1)^2}{2} - \frac{(2-w)^2}{2}, & 1 \le w \le 2 \\ \frac{(3-w)^2}{2}, & 2 \le w \le 3 \\ 0, & otherwise \end{cases}$$

5. Let X be a random variable that takes nonnegative integer values, and is associated with a transform of the form

$$\boldsymbol{M_x(s) = c\,\frac{3 + 4e^{2s} + 2e^{3s}}{3 - e^s}}$$

Where c is some scalar. Find E[X], $\boldsymbol{p_x(1)}$, and E[X|X $\neq$ 0].

Since $1 = M_x(0) = c\,\frac{3+4+2}{3-1}$, so that c = 2/9. We then obtain

$$\mathrm{E[X]} = \frac{dM_x}{ds}(s)|_{s=0} = \frac{2}{9}\,\frac{(3-e^s)(8e^{2s}+6e^{3s}) + e^s(3+4e^{2s}+2e^{3s})}{(3-e^s)^2}|_{s=0} = \frac{37}{18}$$

We now use the identity

$$\frac{1}{3-e^s} = \frac{1}{3}\,\frac{1}{1-e^s\ /\ 3} = \frac{1}{3}\left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots\right)$$

Which is valid as long as s is small enough so that $e^s < 3$. It follows that

$$M_x(s) = \frac{2}{9}\ \frac{1}{3}\ (3 + 4e^{2s} + 2e^{3s})\left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots\right)$$

By identifying the coefficients of e^{0s} and e^s, we obtain

$$p_x(0) = \frac{2}{9}, \quad p_x(1) = \frac{2}{27}$$

Let A = {X $\neq$ 0}, We have

$$p_{X|\{X\in A\}}(k) = \begin{cases} \frac{p_x(k)}{P(A)}, & if\ k \neq 0 \\ 0, & otherwise \end{cases}$$

So that

$$\mathrm{E[X|X \neq 0]} = \sum_{k=1}^{\infty} k p_{X|A}(k) = \sum_{k=1}^{\infty} k p_X(k)/P(A) = \frac{E[X]}{1 - p_X(0)} = \frac{37/18}{7/9} = \frac{37}{14}$$