EE3060 Probability – Proposed questions and answers
GROUP 5
HW#4

Question 1:

$f(x,y)=1$, $0<x<1$, $x<y<x+1$

What is Cov(x,y)?

Answer 1:

$f_X(x)=1$, for $0<x<1$

$f_Y(y)=y$, for $0<y<1$

$=2-y$, for $1<=y<2$

求出 $\mu x=\int f_X(x)dx= 1/2$

$\mu y=\int f_Y(y)dy=1$

又$E(XY) = \int_0^1 \int_x^{x+1} xydydx$

$=\int_0^1 \left(x^2 + \frac{1}{2}x\right) dx$

$=\frac{7}{12}$

则 $Cov(x,y)= E(XY)-\mu x\mu y$

$= \frac{1}{12}$

Q2:

The lifetimes of two light bulbs are modeled as independent and exponential random variables X and Y, with parameters λ and μ, respectively.
The time at which a light bulb first burns out is

$$Z = \min\{X.Y\}.$$

Show that Z is an exponential random variable with parameter $\lambda + \mu$.

For all $z \geq 0$, we have, using the independence of X and Y, and the form of the exponential CDF,

$$\begin{aligned} F_Z(z) &= \mathbf{P}\big(\min\{X, Y\} \leq z\big) \\ &= 1 - \mathbf{P}\big(\min\{X, Y\} > z\big) \\ &= 1 - \mathbf{P}(X > z,\, Y > z) \\ &= 1 - \mathbf{P}(X > z)\mathbf{P}(Y > z) \\ &= 1 - e^{-\lambda z}e^{-\mu z} \\ &= 1 - e^{-(\lambda+\mu)z}. \end{aligned}$$

This is recognized as the exponential CDF with parameter $\lambda + \mu$. Thus, the minimum of two independent exponentials with parameters λ and μ is an exponential with parameter $\lambda + \mu$.

Question 3:

Let X be a random variable that takes nonnegative integer values, and its transform is

$M_X(s) = \frac{13-e^s-6e^{2s}}{15-5e^s-9e^{2s}+3e^{3s}}$. Find $p_X(x)$, $E[X]$, and $var(X)$.

Answer 3:

$$M(s) = \frac{13-e^s-6e^{2s}}{15-5e^s-9e^{2s}+3e^{3s}} = \frac{2}{3-e^s} + \frac{1}{5-3e^{2s}} = \frac{\frac{2}{3}}{1-\frac{1}{3}e^s} + \frac{\frac{1}{5}}{1-\frac{3}{5}e^{2s}}$$

$$= \frac{2}{3}\left[1 + \frac{1}{3}e^s + \frac{1^2}{3}e^{2s} + \frac{1^3}{3}e^{3s} + \cdots\right] + \frac{1}{5}[1 + \frac{3}{5}e^{2s} + \frac{3^2}{5}e^{4s} + \frac{3^4}{5}e^{6s} + \cdots]$$

$$M(s) = \sum_{x=0}^{\infty} e^{sx}p_X(x) = p_X(0)e^{s0} + p_X(1)e^{s1} + p_X(2)e^{s2} + p_X(3)e^{s3} + \cdots$$

$$p_X(x) = \begin{cases} \frac{2}{3}\times\frac{1^x}{3} + \frac{1}{5}\times\frac{3^{\frac{x}{2}}}{5}, & x \in even \\ \frac{2}{3}\times\frac{1^x}{3}, & x \in odd \end{cases}$$

$$E[X] = \frac{dM(s)}{ds}|_{s=0} = \frac{2e^s}{(3-e^s)^2}|_{s=0} + \frac{6e^{2s}}{(5-3e^{2s})^2}|_{s=0} = 2$$

$$E[X^2] = \frac{d^2M(s)}{ds^2}|_{s=0}$$

$$= \frac{(3-e^s)^2\times 2e^s - 2e^s\times(6-2e^s)(-e^s)}{(3-e^s)^4}|_{s=0}$$

$$+ \frac{(5-3e^{2s})^2\times 12e^{2s} - 6e^{2s}\times(10-6e^{2s})\times(-6e^{2s})}{(5-3e^{2s})^4}|_{s=0} = 13$$

$$var(X) = E[X^2] - (E[X])^2 = 13 - 2^2 = 9$$

Question 4:

Problem 10. Let X and Y be independent random variables with PMFs

$$p_X(x) = \begin{cases} 1/3, & \text{if } x = 1,2,3, \\ 0, & \text{otherwise,} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & \text{if } y = 0, \\ 1/3, & \text{if } y = 1, \\ 1/6, & \text{if } y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the PMF of $Z = X + Y$, using the convolution formula.

Answer 4:

$Z=1$: $\frac{1}{3}\times\frac{1}{2} = \frac{1}{6}$

$Z=2$: $\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2} = \frac{5}{18}$

$Z=3$: $\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2} = \frac{1}{3}$

$Z=4$: $\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{3} = \frac{1}{6}$

$Z=5$: $\frac{1}{3}\times\frac{1}{6} = \frac{1}{18}$

$$p_Z(Z) = \begin{cases} \frac{1}{6}, & z=1 \\ \frac{5}{18}, & z=2 \\ \frac{1}{3}, & z=3 \\ \frac{1}{6}, & z=4 \\ \frac{1}{18}, & z=5 \end{cases}$$

Question 5:

Problem 18. Consider four random variables. W. X. Y, Z. with

$$\mathbf{E}[W] = \mathbf{E}[X] = \mathbf{E}[Y] = \mathbf{E}[Z] = 0,$$

$$\text{var}(W) = \text{var}(X) = \text{var}(Y) = \text{var}(Z) = 1.$$

and assume that W. X, Y. Z are pairwise uncorrelated. Find the correlation coefficients $\rho(R.S)$ and $\rho(R.T)$. where $R = W + X$, $S = X + Y$. and $T = Y + Z$.

Answer 5:

$$\text{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\text{cov}(R, S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$