# 10720 EECS 303003 Probability Homework #3 Answer

## Problem1.

**(a)**

First calculate the CDF of X. For x ∈ [0, r], we have

$Fx(x) = P(X \le x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}$ *and* $fx(x) = \frac{2x}{r^2}$ *if* $0 \le x \le r.$

Then, $E[X] = \int_0^r \frac{2x^2}{r^2} dx = \frac{2r}{3}.$ $E[X^2] = \int_0^r \frac{2x^3}{r^2} dx = \frac{r^2}{2}.$

*So* $var(X) = E[X^2] - (E[X])^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}$

**(b)**

$Fs(s) = 0,$ *for* $0 \le s < 1/t$

$Fs(s) = P(S \le s) = P(Alvin's\ hit\ is\ outside\ the\ inner\ circle)$

$= 1 - P(X \le t) = 1 - \frac{t^2}{r^2}$

Also, For $1/t < s$, the CDF of S is given by

$Fs(s) = P(S \le s) = P(X \le t)P(S \le s|X \le t) + P(X > t)P(S \le s|X > t)$

We have $P(X \le t) = \frac{t^2}{r^2},\ P(X > t) = 1 - \frac{t^2}{r^2}$

And since S = 0 when X > t, $P(S \le s|X > t)$ =1

Furthermore,

$$P(S \le s|X \le t) = P\left(\frac{1}{X} \le s \middle| X \le t\right) = \frac{P\left(\frac{1}{s} \le X \le t\right)}{P(X \le t)}$$

$$= \pi t^2 - \frac{\frac{\pi\left(\frac{1}{s}\right)^2}{\pi r^2}}{\frac{\pi t^2}{\pi r^2}} = 1 - \frac{1}{s^{2t^2}}$$

Combine the above equation, we obtain

$P(S \le s) = \frac{t^2}{r^2}(1 - \frac{t^2}{s^2 t^2}) + 1 - \frac{t^2}{r^2} = 1 - \frac{1}{s^2 r^2}$

Collecting the results of the preceding calculations, the CDF of S is

$$Fs(s) = \{\ 0, \quad if\ s < 0$$

$$1 - \frac{t^2}{r^2}, \quad if\ 0 \le s < 1/t$$

$$1-\frac{1}{s^2r^2}, \qquad if\,\frac{1}{t}\leq s$$

Because Fs has a discontinuity at s = 0, the random variable S in not continuous.

## Problem2.

$f(x)=\frac{1}{b-2}; 0<a<x, a+2<x<b$

$F(x)=\{\,0\,;\; x<0$

$\frac{x}{b-2};\; 0<x<a\rightarrow 0\leq x<5$

$\frac{a}{b-2}; a\leq x<a+2\rightarrow 5\leq x<7$

$\frac{x-2}{b-2}; a+2<x<b\rightarrow 7\leq x>22$

$1\,;\; b\leq x\rightarrow x\geq 22$

**(a)**

$F(a+1)=0.25=\frac{a}{b-2}\rightarrow 4a=b-2\rightarrow a=5$

$F(4)=0.2=\frac{4}{b-2}\rightarrow 20=b-2\rightarrow b=22$

**(b)**

$F(8.39)=\frac{8.39-2}{20}=0.3195$

**(c)**

$P(3.01\leq X\leq 9.14)=F(9.14)-F(3.01)=\frac{9.14-2}{20}-\frac{3.01}{20}=0.2065$

## Problem3.

**(a)**

k=1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(x,y)dxdy = \mathrm{k}/\pi\int_{0}^{\infty}\int_{0}^{\infty} e^{\frac{x^2+y^2}{2}}dxdy + \mathrm{k}/\pi\int_{-\infty}^{0}\int_{-\infty}^{0} e^{\frac{x^2+y^2}{2}}dxdy$$

$=\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{\frac{-r^2}{2}} rdrd\theta$=k=1

**(b)**

$fx(x)\ =\int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} f(x,y)dy + \int_{-\infty}^{0} f(x,y)dx$

$=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, -\infty\ < x < \infty$

r.v.Y can follow the same steps. $fy(y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}$

**(c)**

*$f_{XY}(x,y) \neq f_X(x)*f_Y(y)$ thus, X and Y are independent.*

## Problem4.

**(a)**

First find n:

$$\int_0^1\int_0^y n(n-1)(y-x)^2dxdy = 1$$

$$\frac{n(n-1)}{12} = 1, \quad \therefore n = 4$$

As we know that

$$\mathrm{E[Y|X]} = \int_{-\infty}^{\infty} yf(y|x)dy$$

Therefore:

$$f_{y|x} = \frac{f_{xy}}{f_x} = \frac{3\,(y-x)^2}{(1-x)^2}$$

$$\mathrm{E}[Y|\mathrm{X}] = \int_x^1 y\,\frac{3(y-x)^2}{(1-x)^2}dy = \frac{3+x}{4}$$

**(b)**

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X]\ f_x(x)dx = \frac{4}{5}$$

**(c)**

$$\mathrm{E}\big[\ \mathrm{E}[\mathrm{Y}|\mathrm{X}]\big] = \int_{-\infty}^{\infty} E[Y|X]\ f_x(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{y|x}(y|x)\ dy\ f_x(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{y|x}(y|x) f_x(x)dx\,dy = \int_{-\infty}^{\infty} y f_y(y)\ dy = E[Y]$$

## Problem5.

r.v.X,Y ~i.i.d E(λ)

$f_X(x) = \lambda e^{-\lambda x}u(x)$

$f_Y(y) = \lambda e^{-\lambda y}u(y)$

$f_{XY}(x,y) = \lambda^2 e^{-\lambda(x+y)}u(x)u(y)$

$f_Z(z) = \int_{-\infty}^{\infty} f(x = z - y, y)dy = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda z}u(z-y)u(y)dy = \int_0^z \lambda^2 e^{-\lambda z}dy$

$= \lambda^2 e^{-\lambda z}$, z>0

0 , o.w.

## Problem6.

*P(X=1) = P*

*P(X=-1) = 1-P*

*Y = X + M → M = Y – X*

$$f(m) = \left(\frac{1}{2}\right)\alpha e^{-\alpha|m|}, -\infty < m < \infty$$

$$\mathrm{f(y)} = pf(y|x=1) + (1-p)f(y|x=-1)$$

$= \left(\frac{p}{2}\right)e^{-\alpha|y-1|} + \left(\frac{1-p}{2}\right)e^{-\alpha|y+1|}$, $-\infty < y < \infty$