Probability_2

Question 1:

Suppose there is a box with n “odd number labeled” balls and n “even number labeled” balls. Henry decides to pick 2 or more balls from the box. After each pick, he adds the number on the newly picked ball to the number on the previously picked ball, and replaces the ball. He repeats the process until the sum of the numbers added is an even number. Find the expected value of the number of times Henry picks a ball. (Note: Henry decides to pick 2 or more balls, meaning if the first ball is even, he continues to pick until the sum of the numbers on the 2 or more balls is even.)

Answer 1:

Let X be the random variable for the number of times Henry picks a ball.
Let $even_i$ be the event in which the i^{th} ball chosen has an even number on it.
Let odd_i be the event in which the i^{th} ball chosen has an odd number on it.
By total expectation theorem,

$$E[X] = \frac{1}{2}E[X|even_1] + \frac{1}{2}E[X|odd_1]$$

Now we look at $E[X|even_1]$, apply total expectation theorem,

$$E[X|even_1] = \frac{1}{2}E[X|even_1 \cap even_2] + \frac{1}{2}E[X|even_1 \cap odd_2]$$

$$= \frac{1}{2}\times 2 + \frac{1}{2}E[X|even_1 \cap odd_2]$$

We also know that

$$E[X|even_1 \cap odd_2] = E[X|odd_1] + 1$$

Why is this true?
Let Y be a random variable that has the property $Y = X|even_1 \cap odd_2$
Let Y' be a random variable that has the property $Y' = X|odd_1$

Y	3	4	5	6	7
probability	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$

Y'	2	3	4	5	6
probability	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$

Therefore,

$$Y = Y' + 1$$

$$E[Y] = E[Y'] + 1$$

$$E[X|even_1 \cap odd_2] = E[X|odd_1] + 1$$

So,

$$E[X|even_1] = \frac{1}{2}\times 2 + \frac{1}{2}(E[X|odd_1] + 1)$$

We also know that

$$E[X|odd_1] = \frac{1}{2}E[X|odd_1 \cap odd_2] + \frac{1}{2}E[X|odd_1 \cap even_2]$$

$$= \frac{1}{2}\times 2 + \frac{1}{2}(E[X|odd_1] + 1)$$

Why is $E[X|odd_1 \cap even_2] = (E[X|odd_1] + 1)$?

Let Z be a random variable that has the property $Z = X|odd_1 \cap even_2$

Let Z' be a random variable that has the property $Z' = X|odd_1$

Z	3	4	5	6	7
probability	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$

Z'	2	3	4	5	6
probability	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$

Therefore,

$$Z = Z' + 1$$

$$E[Z] = E[Z'] + 1$$

$$E[X|odd_1 \cap even_2] = E[X|odd_1] + 1$$

By rearranging the next equation,

$$E[X|odd_1] = \frac{1}{2}\times 2 + \frac{1}{2}(E[X|odd_1] + 1)$$

We get $E[X|odd_1] = 3$

Plug this back to

$$E[X|even_1] = \frac{1}{2}\times 2 + \frac{1}{2}(E[X|odd_1] + 1)$$

We get $E[X|even_1] = 3$

Finally, we go back to the initial equation

$$E[X] = \frac{1}{2}E[X|even_1] + \frac{1}{2}E[X|odd_1]$$

We get $E[X] = 3$

Ans: $E[X] = 3$

Question 2:

A fair die is tossed 6 times. What is the expected number of times T that two consecutive pionts occur? For example, if the outcome is 111223 then T = 3.

Answer 2:

Let Ti be the random variable that takes value 1 if tosses i, i + 1 are the same, and 0 if not. Then T = T1 + T2 +···+ T5. By the linearity of expectation E[T] = E[T1] +···+ E[T5]. Each Ti takes value 1 with probability 6/36 = 1/6. Therefore E[T] = 5*(1/6) = 5/6.

Question 3:

Let X be a discrete random variable with PMF $p_X(x)$ given by

$$p_X(x)=\begin{cases} (\frac{1}{2})^{x+3}, & if\ x = -3,-2,-1,0,1,2,3 \\ 0, & otherwise \end{cases}$$

Let $Y = X^2$, and let p_Y(y) denote PMT of Y. Find $p_Y(1)$ $p_Y(4)$, $p_Y(5)$.

Answer 3:

Because $Y = X^2$, now y becomes 0, 1, 4, 9.

And, $p_Y(y=1)$ = P[x = 1] + P[x = -1]

$$= (\frac{1}{2})^4 + (\frac{1}{2})^2$$

And, $p_Y(y=4)$ = P[x = 2] + P[x = -2]

$$= (\frac{1}{2})^5 + (\frac{1}{2})^1$$

And, $p_Y(y=5)$ = 0. Because 5 is not one of y.

Question 4:

assume that the probability that an engine being broken in an hour is 0.02, what's the probability that the engine is still working after 2 hours? What's the expected value of the engine working hour?

Answer 4:

1. P(x>3)=1-p(2)-p(1)=1-0.02-0.98*0.02=0.9604
2. 1/0.02=50

Question 5:

Recently, a seasonal flu outbreaks within Electrical Engineering Department. The probability for each person to get a flu is p; however, the students who has boy/girlfriend will double the probability of getting the flu. Supposed that there are 2m people in EE Department, and there are b couples (2b people), find the PMF, Expectation and Variance for all people in EE. (Don't need to simplify the answer)

Answer 5:

Let X = number of single people get flu.

Y = number of coupled people get flu.

Z = number of people get flu or not.

We can know that X,Y are independent, and Z=X+Y.

P(Z)=P(single)P(get flu|single)+P(coupled)P(get flu|coupled)=P(X)+P(Y)=$\frac{2m-2b}{2m} * p + \frac{2b}{2m} * 2p$

E(Z)=E(X)+E(Y)=2m*P(Z)=$(2m-2b) * p + 2b * 2p$

Var(Z)=Var(X)+Var(Y)={ $[1-E(X)]^2$*p+$[0-E(X)]^2$*(1-p) } + {$[1-E(Y)]^2$*2p+$[0-E(Y)]^2$*(1-2p) }