Problem 1 - solution

(a) Let L be the duration of the match. If Fischer wins a match consisting of L games, then $L-1$ draws must first occur before he wins. Summing over all possible lengths, we obtain

$$\mathbf{P}(\text{Fischer wins}) = \sum_{l=1}^{10} (0.3)^{l-1}(0.4) = 0.571425.$$

(b) The match has length L with $L < 10$, if and only if $(L-1)$ draws occur, followed by a win by either player. The match has length $L = 10$ if and only if 9 draws occur. The probability of a win by either player is 0.7. Thus

$$p_L(l) = \mathbf{P}(L = l) = \begin{cases} (0.3)^{l-1}(0.7), & l = 1, \ldots, 9, \\ (0.3)^9, & l = 10, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2 - solution

(a) Let X be the number of packets stored at the end of the first slot. For k<b, the probability that X=k is the sane as the probability that k packets are generated by the source:.

$$\text{px(k)} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0,1,2,,,,, b-1$$

while

$$\text{px(x)} = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & 0 \le x < b \\ \sum_{x=b}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = 1 - \sum_{x=0}^{b-1} e^{-\lambda}\frac{\lambda^x}{x!}, & x = b \\ 0, & x > b \end{cases}$$

(b)

Let Y be the number of number of packets stored at the end of the second slot. Since $\min\{X, c\}$ is the number of packets transmitted in the second slot, we have $Y = X - \min\{X, c\}$. Thus,

$$p_Y(0) = \sum_{k=0}^{c} p_X(k) = \sum_{k=0}^{c} e^{-\lambda}\frac{\lambda^k}{k!},$$

$$p_Y(k) = p_X(k+c) = e^{-\lambda}\frac{\lambda^{k+c}}{(k+c)!}, \qquad k = 1, \ldots, b-c-1,$$

Problem 3 - solution

(a) The scalar a must satisfy

$$1 = \sum_x p_X(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2,$$

so

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = 28.$$

We also have $\mathbf{E}[X] = 0$ because the PMF is symmetric around 0.

(b) If $z \in \{1, 4, 9\}$, then

$$p_Z(z) = p_X(\sqrt{z}) + p_X(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \frac{z}{14}.$$

Otherwise $p_Z(z) = 0$.

(c) $\mathrm{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = 7.$

(d) We have

$$\begin{aligned}
\mathrm{var}(X) &= \sum_x (x - \mathbf{E}[X])^2 p_X(x) \\
&= 1^2 \cdot \big(p_X(-1) + p_X(1)\big) + 2^2 \cdot \big(p_X(-2) + p_X(2)\big) + 3^2 \cdot \big(p_X(-3) + p_X(3)\big) \\
&= 2 \cdot \frac{1}{28} + 8 \cdot \frac{4}{28} + 18 \cdot \frac{9}{28} \\
&= 7.
\end{aligned}$$

Problem 4 - solution

Let X be the total number of tosses.

(a) For each toss after the first one, there is probability 1/2 that the result is the same as in the preceding toss. Thus, the random variable X is of the form $X = Y + 1$, where Y is a geometric random variable with parameter $p = 1/2$. It follows that

$$p_X(k) = \begin{cases} (1/2)^{k-1}, & \text{if } k \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{E}[X] = \mathbf{E}[Y] + 1 = \frac{1}{p} + 1 = 3.$$

We also have

$$\text{var}(X) = \text{var}(Y) = \frac{1-p}{p^2} = 2.$$

(b) If $k > 2$, there are $k - 1$ sequences that lead to the event $\{X = k\}$. One such sequence is $H \cdots HT$, where $k-1$ heads are followed by a tail. The other $k-2$ possible sequences are of the form $T \cdots TH \cdots HT$, for various lengths of the initial $T \cdots T$ segment. For the case where $k = 2$, there is only one (hence $k - 1$) possible sequence that leads to the event $\{X = k\}$, namely the sequence HT. Therefore, for any $k \geq 2$,

$$\mathbf{P}(X = k) = (k-1)(1/2)^k.$$

It follows that

$$p_X(k) = \begin{cases} (k-1)(1/2)^k, & \text{if } k \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{E}[X] = \sum_{k=2}^{\infty} k(k-1)(1/2)^k = \sum_{k=1}^{\infty} k(k-1)(1/2)^k = \sum_{k=1}^{\infty} k^2(1/2)^k - \sum_{k=1}^{\infty} k(1/2)^k = 6-2 = 4.$$

We have used here the equalities

$$\sum_{k=1}^{\infty} k(1/2)^k = \mathbf{E}[Y] = 2,$$

and

$$\sum_{k=1}^{\infty} k^2(1/2)^k = \mathbf{E}[Y^2] = \text{var}(Y) + \big(\mathbf{E}[Y]\big)^2 = 2 + 2^2 = 6,$$

where Y is a geometric random variable with parameter $p = 1/2$.

Problem 5 - solution

Let X_i be the random variable taking the value 1 or 0 depending on whether the first partner of the ith couple has survived or not. Let Y_i be the corresponding random variable for the second partner of the ith couple. Then, we have $S = \sum_{i=1}^{m} X_i Y_i$, and by using the total expectation theorem,

$$\begin{aligned}
\mathbf{E}[S \mid A = a] &= \sum_{i=1}^{m} \mathbf{E}[X_i Y_i \mid A = a] \\
&= m\mathbf{E}[X_1 Y_1 \mid A = a] \\
&= m\mathbf{E}[Y_1 = 1 \mid X_1 = 1,\ A = a]\mathbf{P}(X_1 = 1 \mid A = a) \\
&= m\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\ A = a)\mathbf{P}(X_1 = 1 \mid A = a).
\end{aligned}$$

We have

$$\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\ A = a) = \frac{a-1}{2m-1}, \qquad \mathbf{P}(X_1 = 1 \mid A = a) = \frac{a}{2m}.$$

Thus

$$\mathbf{E}[S \mid A = a] = m\,\frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}.$$

Note that $\mathbf{E}[S \mid A = a]$ does not depend on p.