EE 3060 **Midterm Exam #3**

Note: If $X \sim \mathcal{G}(\alpha, \lambda)$, then

$$f_X(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. (10%) Let the probability density function of a random variable X be given by

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$Y = \begin{cases} X, & \text{if } X \leq 1, \\ \frac{1}{X}, & \text{if } X > 1. \end{cases}$$

Find the probability density function $f_Y(y)$ of Y.

2. (a) (7%) Let $X \sim N(\mu, \sigma^2)$ and $Y = \sqrt{|X|}$. Find the probability density function $f_Y(y)$ of Y.

(b) (8%) Let $X_i \sim N(\mu_i, \sigma^2)$, $i = 1, 2, \ldots, n$, be independent random variables and $Y = \sum_{i=1}^{n}(X_i - \mu_i)^2$. Find the probability density function $f_Y(y)$ of Y.

3. (a) (10%) Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent random variables. What is the probability that the quadratic equation $t^2 + Xt + 2Y = 0$ has two real roots?

(b) (10%) Let X and Y be the x-coordinate and the y-coordinate, respectively, of a point randomly selected from the region $R = \{(x,y) : |x| + |y| \leq 1\}$. Find $f_X(x)$ and $f_{X|Y}(x|y)$.

4. Let the joint probability density function of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}e^{-x}, & \text{if } x \geq 0 \text{ and } |y| \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (7%) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

(b) (8%) Find $E[Y|X = x]$ and $Var[Y|X = x]$.

$Y = \ln V$ $U - \ln V$

5. (10%) Let $X \sim \mathcal{E}(\lambda)$ and $Y \sim \mathcal{E}(\lambda)$ be independent random variables, and let $U = X + Y$ and $V = e^X$. Find $f_{U,V}(u,v)$.

6. (10%) Michael enters a bank with n tellers. All of the tellers are busy serving customers and there is exactly one queue being served by all tellers, with one customer ahead of Michael waiting to be served. If the service time of a customer is an exponential random variable with parameter λ, what can you say about Michael's waiting time in the queue before he is served by one of the tellers? n

7. (a) (10%) Let $X \sim \mathcal{E}(\lambda)$ and $Y \sim \mathcal{E}(\lambda)$ be independent random variables. Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent.

(b) (10%) Let $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ be independent random variables. Find $f_{X_{(1)},X_{(2)}}(x_1, x_2)$ and $E[X_{(1)}]$.

$X_{(1)} = n [F(x)] [1-F(x)]^{n-1}$ $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

Good luck!