EE 3060 **Midterm Exam #1**

1. (7%) What is a probability space?

2. Let $A = \{1, 2\}$ and $\mathcal{P}(A)$ be the power set of A. For the experiment that two distinct subsets of A are chosen randomly from $\mathcal{P}(A)$, describe the following events:
(a) (2%) The certain event, i.e., the sample space of this experiment.
(b) (2%) The event that the intersection of the two chosen subsets is empty.
(c) (2%) The event that the two chosen subsets are complements of each other.
(d) (2%) The event that one of the chosen subsets contains more elements than the other.

3. (10%) An integer is chosen randomly from the set $\{1, 2, \ldots, 1000\}$. What is the probability that it is divisible by 4 but neither by 5 nor by 7?

4. (10%) (Borel-Cantelli Lemma) Let $\{A_1, A_2, \ldots\}$ be a sequence of events. Show that if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) = 0$. (Hint: Let $B_m = \cup_{n=m}^{\infty} A_n$.)
Remark: From Exercise 1.2.19 (HW #1), we know that $\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$ is the event that infinitely many of the A_i's occur. As such, the Borel-Cantelli Lemma says that if the series $\sum_{n=1}^{\infty} P(A_n)$ converges, then the probability that infinitely many of the A_i's occur is 0.

5. (a) (7%) Suppose that A_i, $i = 1, 2, \ldots, n$, are n events in a probability space $(\Omega, \mathcal{A}, P)$ and $P(A_i) = 1$ for $i = 1, 2, \ldots, n$. Show that $P(\cap_{i=1}^{n} A_i) = 1$. (Hint: Prove by induction on n.)
(b) (8%) Show that part (a) is not true for an infinite number of events by giving a counterexample that $\{B_t : 0 < t < 1\}$ is a collection of an infinite number of events with $P(B_t) = 1$ for all $0 < t < 1$, but $P(\cap_{t \in (0,1)} B_t) \neq 1$. (Hint: Consider the experiment of choosing a point randomly from the interval $(0, 1)$.)

6. (15%) In a lottery, players pick 6 different integers from the set $\{1, 2, \ldots, 49\}$, the order of selection being irrelevant. The lottery commission then randomly selects 6 of the 49 numbers as the winning numbers. What is the probability that at least two consecutive integers are selected among the winning numbers?

7. (7%) (a) Suppose that n distinguishable balls are randomly placed into n distinguishable boxes. What is the probability that each box will be occupied?
(8%) (b) Suppose that k indistinguishable balls are randomly placed into n distinguishable boxes. If $k \geq n - 1$, then what is the probability that exactly one box remains empty?

8. A train consists of n cars. Each of the m passengers ($m \geq n$) chooses a car randomly to ride in.
(10%) (a) What is the probability that there will be at least one passenger in each car?
(10%) (b) If $r < n$, then what is the probability that exactly r cars remain unoccupied?

Good luck!