EE 3060 **Final Exam**

1. (a) (5%) What is a probability space $(\Omega, \mathcal{A}, P)$?

(b) (10%) Let $A_1, A_2, \ldots$ be an increasing sequence of events of a probability space $(\Omega, \mathcal{A}, P)$. Show that $P(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n)$.

(c) (5%) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $B \in \mathcal{A}$ such that $P(B) > 0$. Let $P(\cdot|B) : \mathcal{A} \to R$ be given by $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Show that $(\Omega, \mathcal{A}, P(\cdot|B))$ is a probability space.

2. (a) (5%) What is a measurable function f from a measurable space $(\Omega_1, \mathcal{A}_1)$ to another measurable space $(\Omega_2, \mathcal{A}_2)$?

(b) (10%) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Show that $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random vector of the probability space $(\Omega, \mathcal{A}, P)$ if and only if X_i is a random variable of the probability space $(\Omega, \mathcal{A}, P)$ for all $i = 1, 2, \ldots, n$.

(c) (5%) Let X be a random variable of a probability space $(\Omega, \mathcal{A}, P)$ and let $P_X : \mathcal{B}_R \to R$ be given by $P_X(B) = P(X^{-1}(B))$ for all $B \in \mathcal{B}_R$. Show that $(R, \mathcal{B}_R, P_X)$ is a probability space.

3. Let the joint probability density function of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}x^3 e^{-x(y+1)}, & \text{if } x > 0 \text{ and } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10%) Find the means and variances of X and Y.

(b) (10%) Find the covariance and correlation coefficient between X and Y. Are X and Y positively correlated, negatively correlated, or uncorrelated?

4. (10%) Let X and Y be jointly normal random variables, and let $U = \frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$ and $V = \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}$. Show that U and V are independent random variables.

5. (10%) From a distribution with mean μ and variance σ^2, a random sample $\{X_1, X_2, \ldots, X_n\}$ of size n is chosen. How large should the sample size n be so that the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is discrepant from the mean μ by less than two standard deviations of the distribution with a probability of at least 0.98?

6. (10%) Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. random variables with the same mean $\mu < \infty$, and let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Using the weak law of large numbers, show that S_n grows at rate n, i.e.,

$$\lim_{n\to\infty} P(n(\mu - \epsilon) \leq S_n \leq n(\mu + \epsilon)) = 1, \text{ for all } \epsilon > 0.$$

7. (10%) Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. Poisson random variables with parameter 1. Using the central limit theorem, show that

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{e^{-n} n^i}{i!} = \frac{1}{2}.$$

Good luck!

$$-\frac{1}{x}\left(y-\frac{1}{2}\right)e^{-xy}\Big|_0^\infty + \int_0^\infty \frac{1}{x}e^{-xy}\,dy$$

$$= \frac{1}{2x} + -\frac{1}{x^2}e^{-xy}\Big|_0^\infty = \frac{1}{2x} + \frac{1}{x^2}$$

$$= \frac{1}{x}\left(\frac{1}{x}+\frac{1}{2}\right)$$