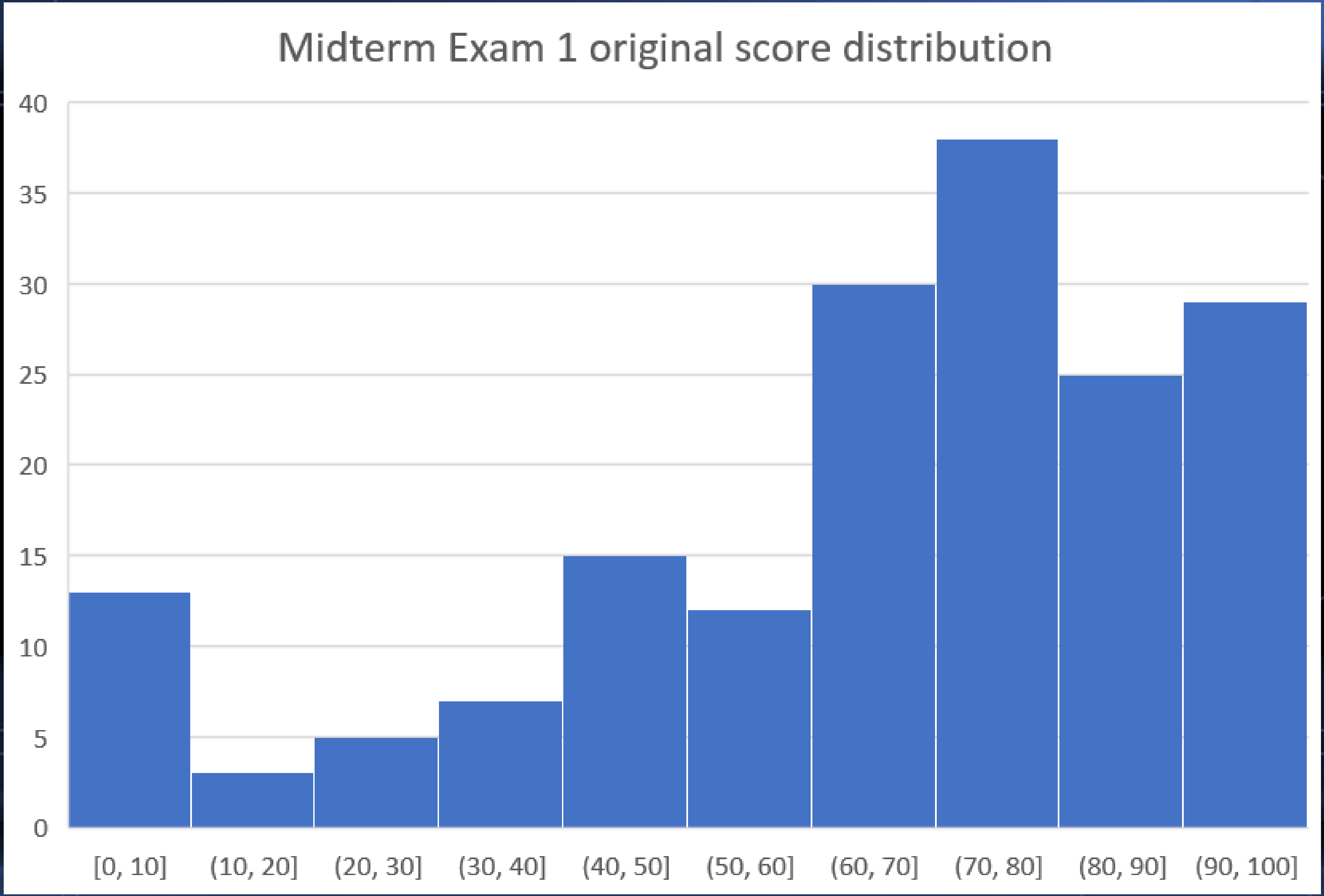
Course announcement

- Final midterm score will be posted on eLearn today. If you have any questions about score, please directly contact me (yhlin@phys.nthu.edu.tw).
- Midterm warning will be sent out during weekend


Midterm Exam 1 original score distribution
40
35
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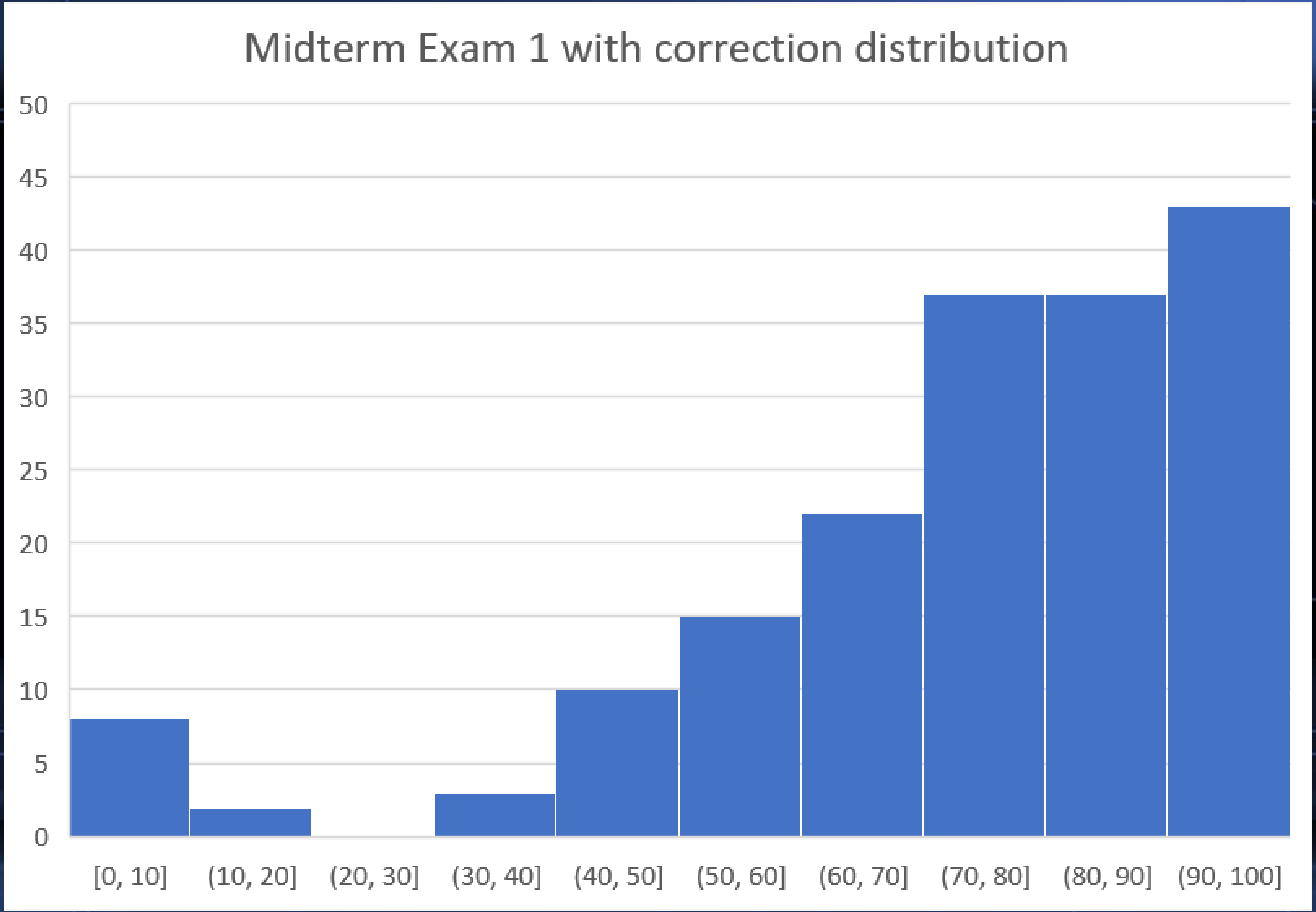
Midterm Exam 1 with correction distribution
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(20, 30]
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(60, 70]
(70, 80]
(80, 90]
(90, 100]

8	11/1(Tue.)	**Many Particles Motion and Rotation:** rotation
8	11/4(Fri.)	**Many Particles Motion and Rotation:** torque & angular momentum
9	11/8(Tue.)	**Oscillation and Waves:** simple harmonic oscillation
9	11/11(Fri.)	**Oscillation and Waves:** damped and forced oscillation (Homework3)

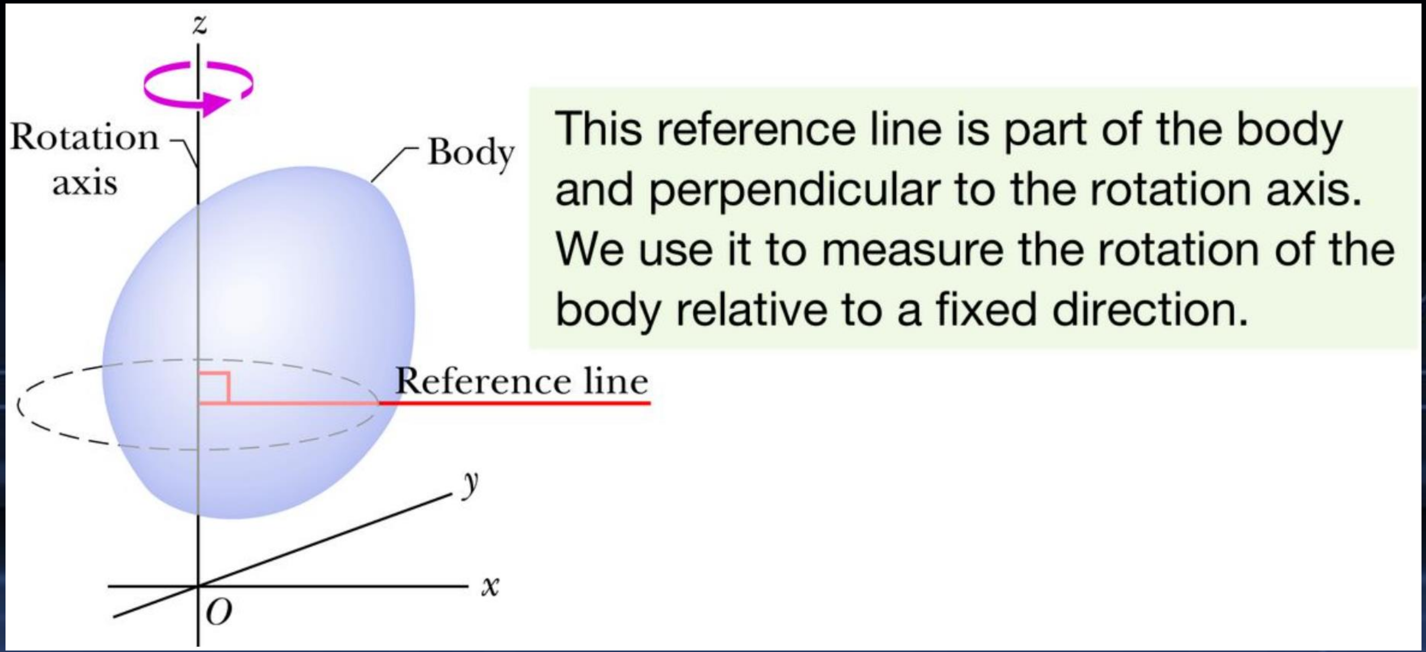
GENERAL PHYSICS B1
MANY PARTICLES MOTION & ROTATION

Rotation, Torque, and Angular Momentum

2022/04/12

The axis of rotation

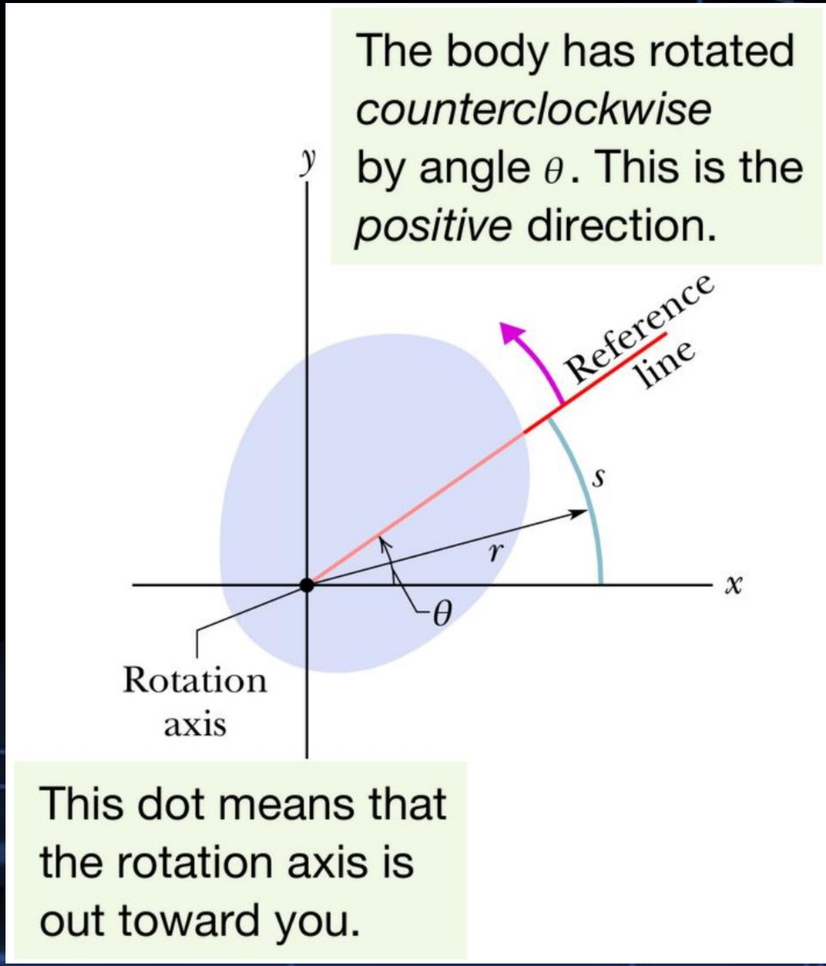
- In the following figure, a rigid body of arbitrary shape in rotation about a fixed axis, called the axis of rotation or the rotation axis.



Angular position

- In pure rotation (angular motion), every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval.
- The angular position θ is measured relative to the positive direction of the x axis.

$$\theta = \frac{s}{r}$$

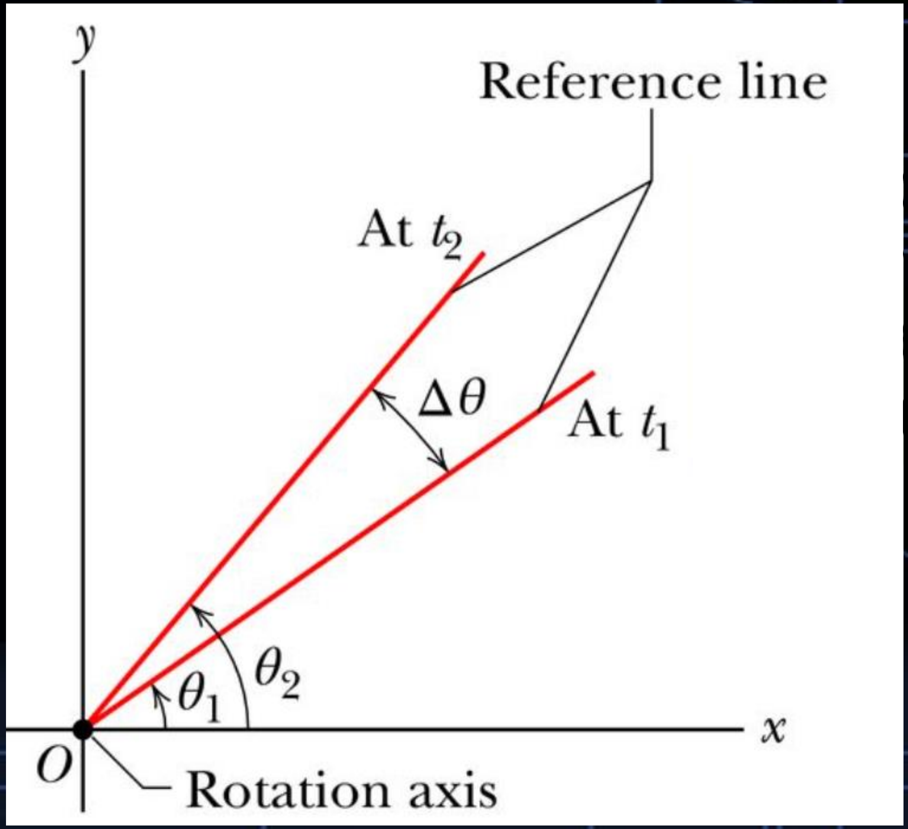


Angular displacement

If the body rotates about the rotation, changing the angular position of the reference line from θ1 to θ2, the body undergoes an angular displacement Δθ given by:

$$\Delta\theta = \theta_2 - \theta_1$$

This definition of angular displacement holds not only for the rigid body as a whole but also for every particle within that body.



Rotation with constant angular acceleration

Linear Equation	Missing Variable		Angular Equation
$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Relation between linear and angular variable

Conversion between linear motion and rotation:

$$s = \theta r$$

$$v = \omega r$$

$$a_t = \alpha r$$

$$a_r = \frac{v^2}{r} = \omega^2 r$$

Rotational inertia (or moment of inertia)

In the rotating rigid body the rotational velocity is the same for every point. Thus we can rewrite:

$$K = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

We call that quantity the rotational inertia (or moment of inertia) I of the body with respect to the axis of rotation:

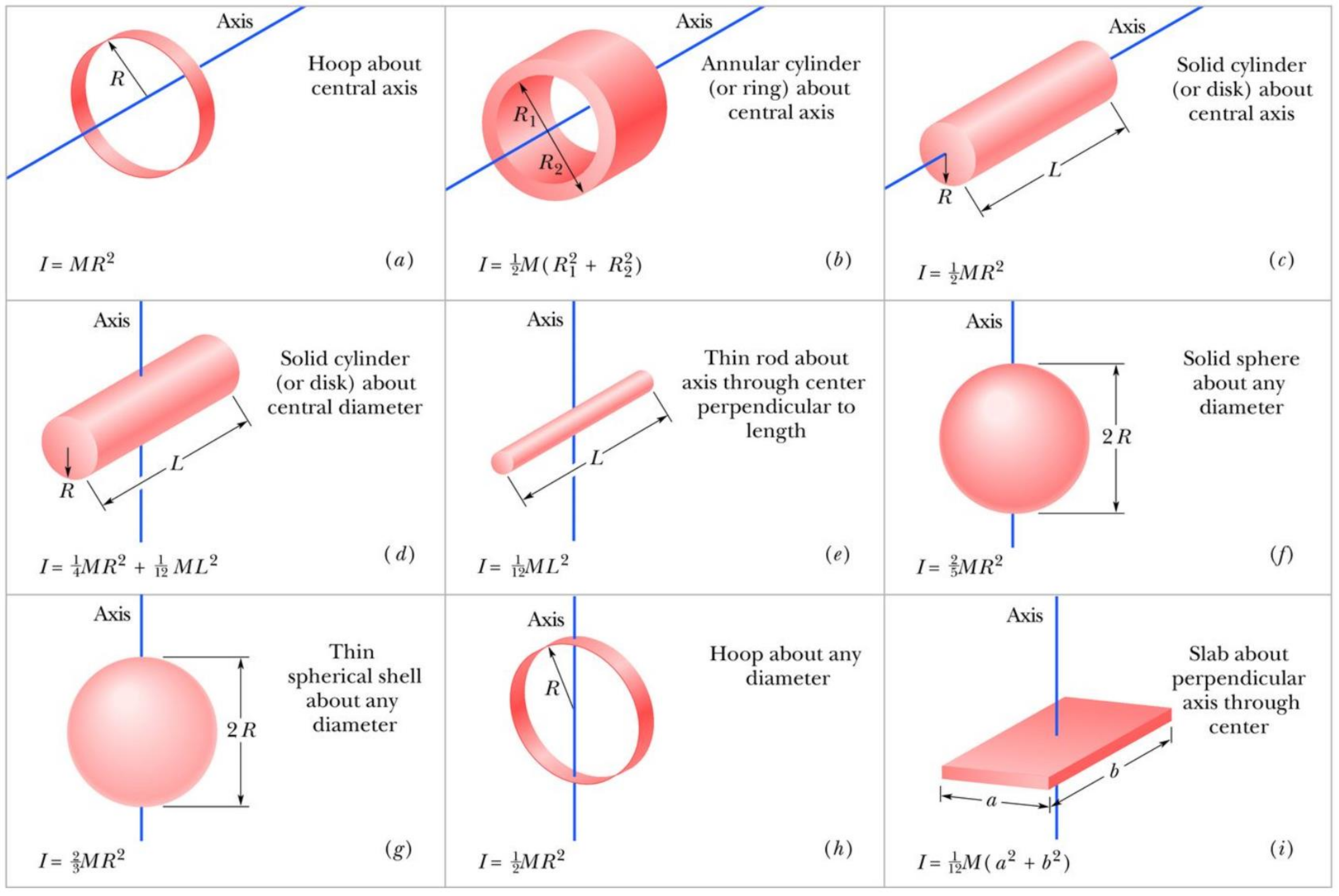
$$I = \sum m_i r_i^2$$

And thus the kinetic energy is $K = \frac{1}{2} I \omega^2$

Calculating the rotational inertia

- If a rigid body consists of a few particles, we can calculate its rotational inertia about a given rotation axis with $I = \sum m_i r_i^2$.
- If a rigid body consists of a great many adjacent particles, we replace the sum with an integral and define the rotational inertia of the body as

$$I = \int r^2 dm$$



Axis
R
Hoop about central axis
$I = MR^2$
(a)
Axis
R_1
R_2
Annular cylinder (or ring) about central axis
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
(b)
Axis
L
R
Solid cylinder (or disk) about central axis
$I = \frac{1}{2}MR^2$
(c)
Axis
L
R
Solid cylinder (or disk) about central diameter
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
(d)
Axis
L
Thin rod about axis through center perpendicular to length
$I = \frac{1}{12}ML^2$
(e)
Axis
2R
Solid sphere about any diameter
$I = \frac{2}{5}MR^2$
(f)
Axis
2R
Thin spherical shell about any diameter
$I = \frac{2}{3}MR^2$
(g)
Axis
R
Hoop about any diameter
$I = \frac{1}{2}MR^2$
(h)
Axis
a
b
Slab about perpendicular axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
(i)

Today's topic

- Parallel-Axis Theorem
- Torque and Newton's second law of rotation
- Conservation of energy and angular momentum

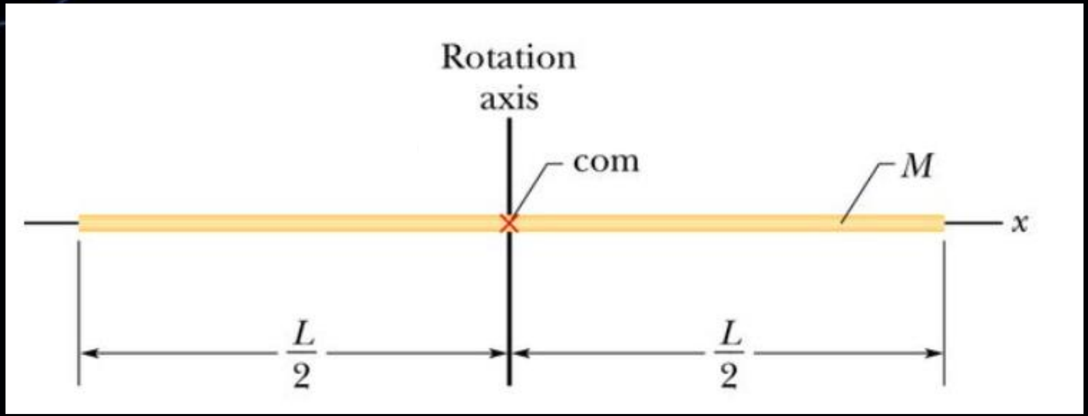
Parallel-Axis Theorem

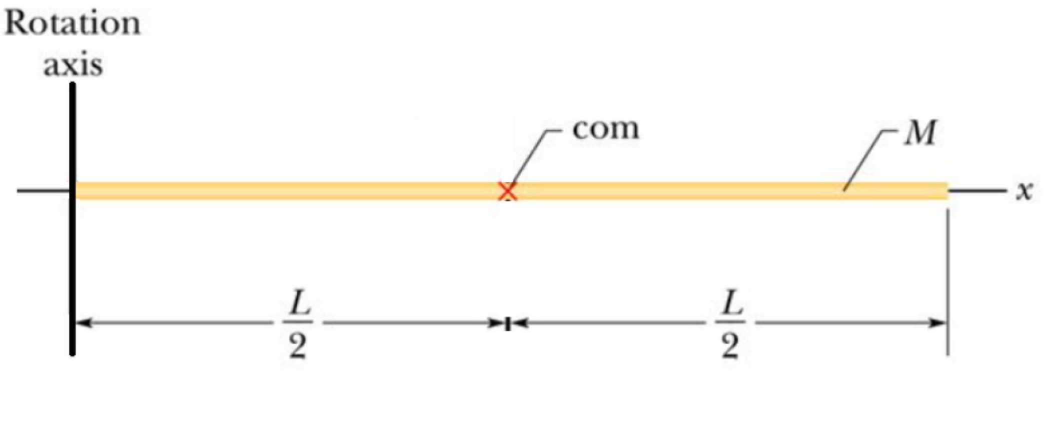
- For calculating rotational inertia, directly calculation through $I = \int r^2 dm$ can work.
- Assuming we know the rotational inertia I_{com}, where the rotational axis is through the body's center of mass. Then, the rotational inertia I about a rotational axis parallel to the rotational axis through COM is:

$$I = I_{COM} + Mh^2$$

where M is the mass of the body and h is the distance that we have shifted the rotation axis from being through the com.

Example of parallel theorem





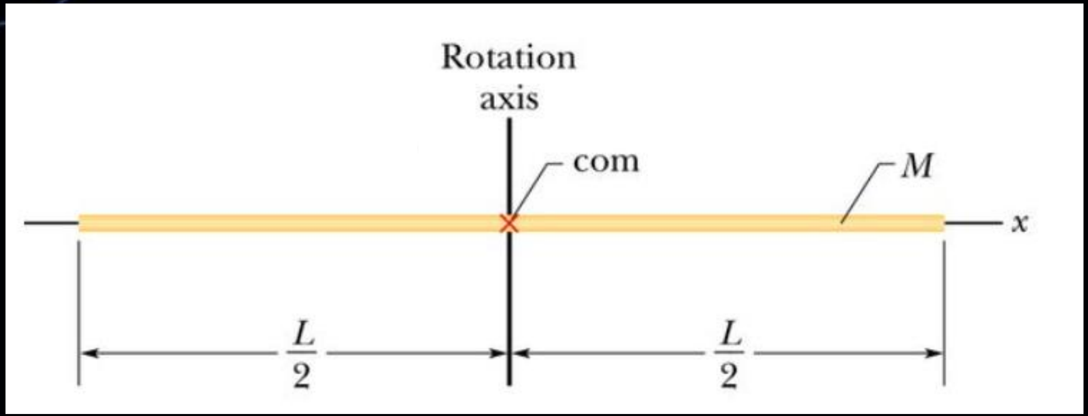
$$I_1 = I_{COM} = \frac{1}{12} ML^2$$

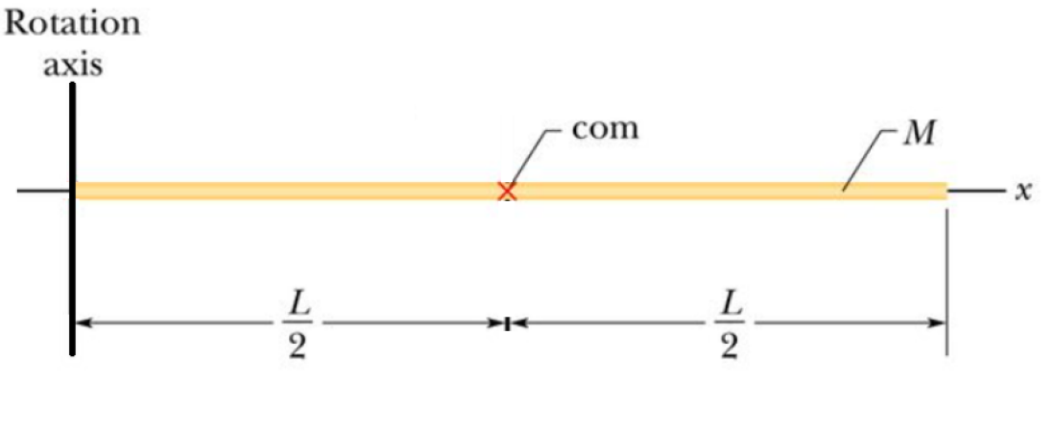
As we calculated last time

$$I_2 = \int_{x=0}^{x=L} x^2 \frac{M}{L} dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{x=0}^{x=L}$$

$$= \frac{1}{3} ML^2$$

Example of parallel theorem





$$I_1 = I_{COM} = \frac{1}{12}ML^2$$

As we calculated last time

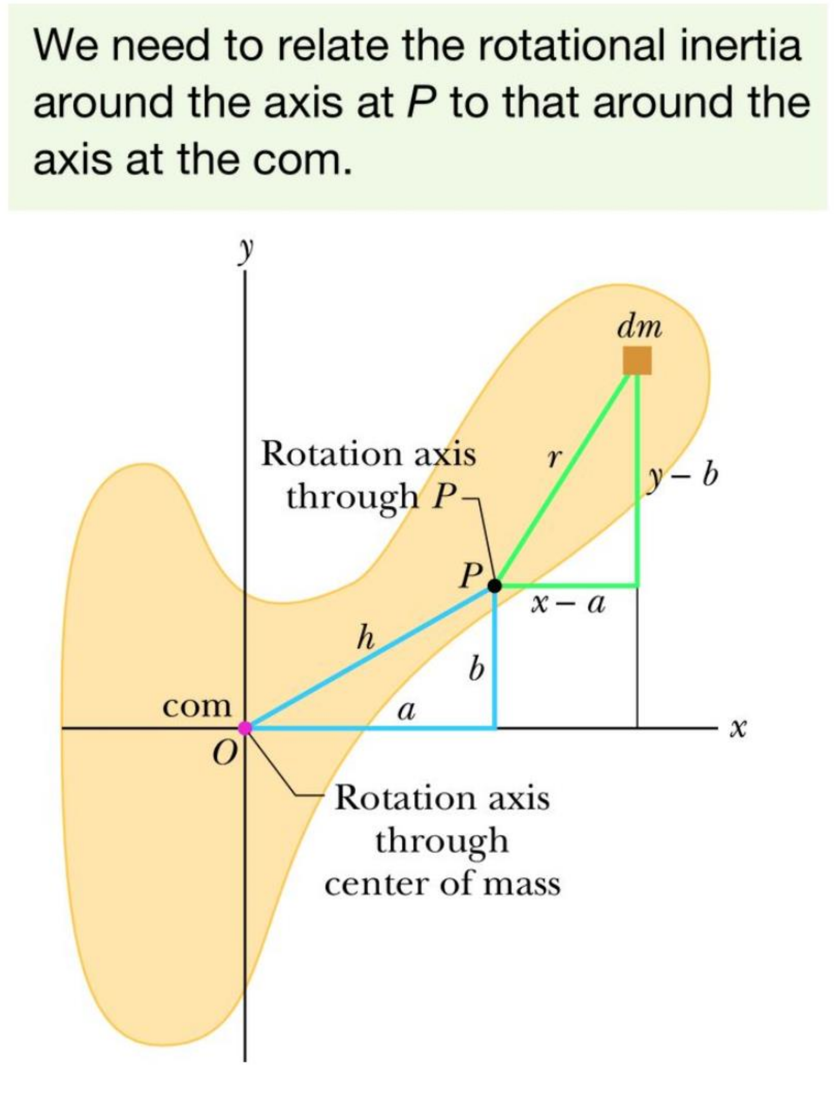
$$I_2 = I_{COM} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

$$= \frac{1}{3}ML^2$$

Proof of the parallel theorem

- Let O be the center of mass of the arbitrarily shaped body shown in cross section. Place the origin of the coordinates at O. Consider an axis through O perpendicular to the plane of the figure, and another axis through point P parallel to the first axis. Let the x and y coordinates of P be a and b.



Proof of the parallel theorem

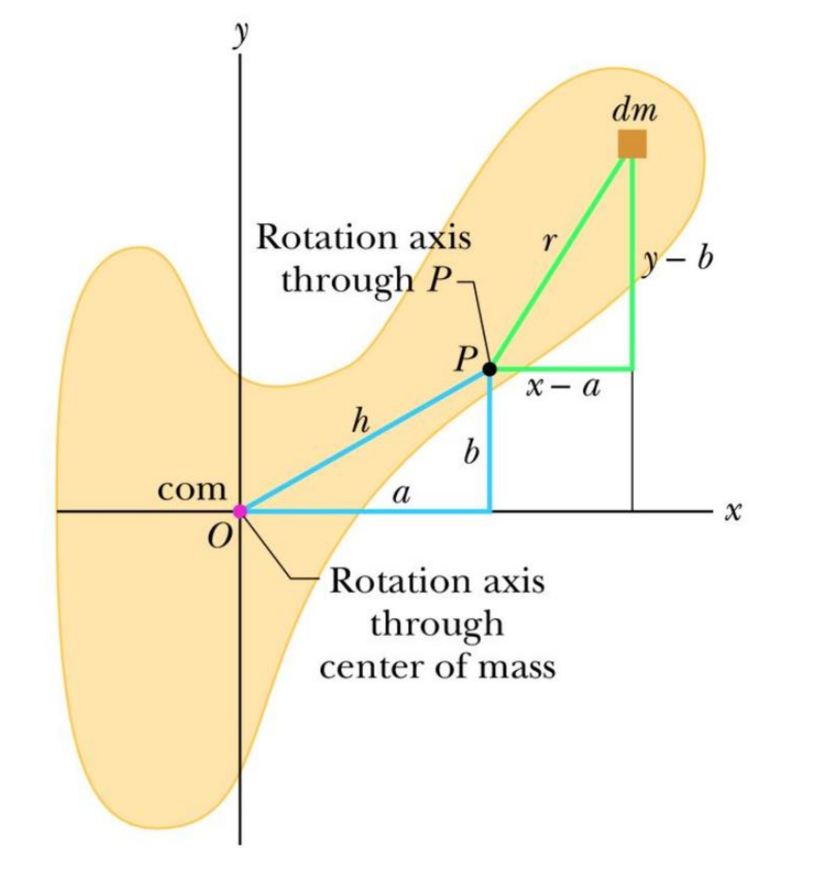
- Let dm be a mass element with the general coordinates x and y. The rotational inertia of the body about the axis through P is

$$I = \int r^2 dm = \int [(x-a)^2 + (y-b)^2] dm$$

$$= \underbrace{\int (x^2+y^2) dm}_{= I_{COM}} - 2\underbrace{\int (ax+by) dm}_{= 0} + \underbrace{\int (a^2+b^2) dm}_{= Mh^2}$$

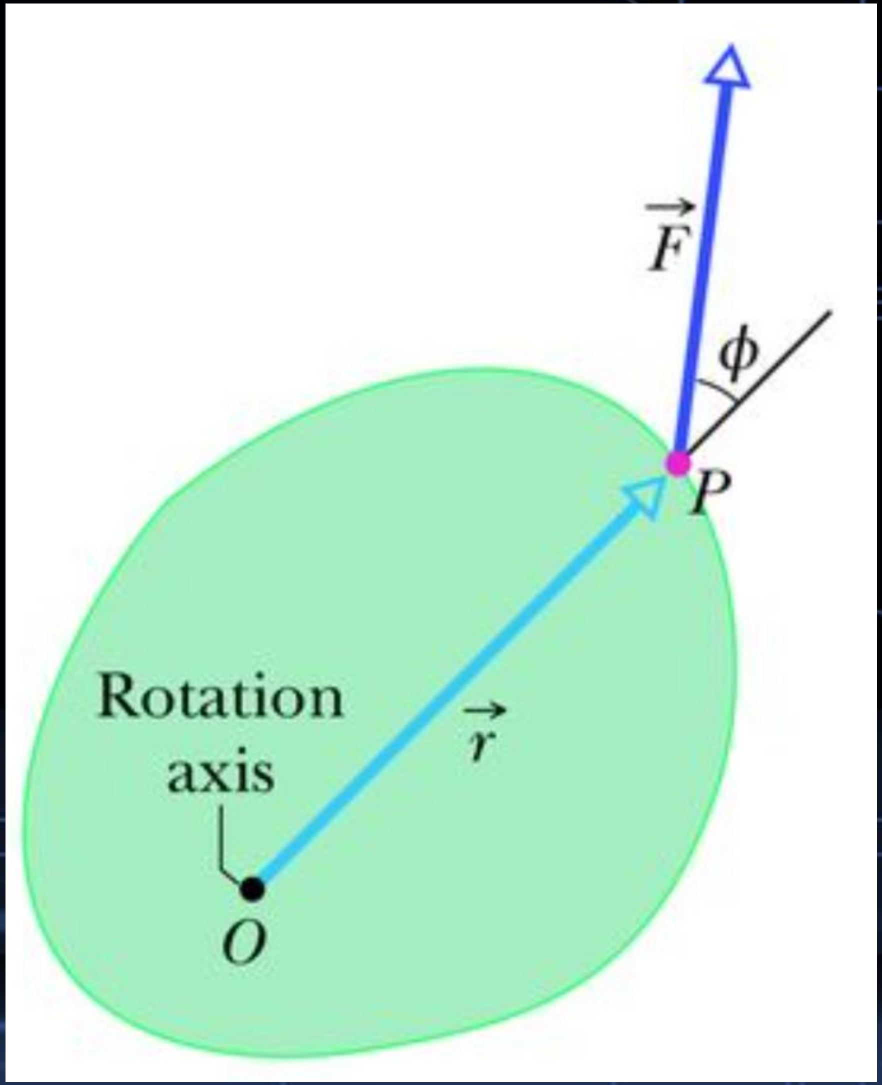
$$I = I_{COM} + Mh^2$$

We need to relate the rotational inertia around the axis at *P* to that around the axis at the com.



Torque

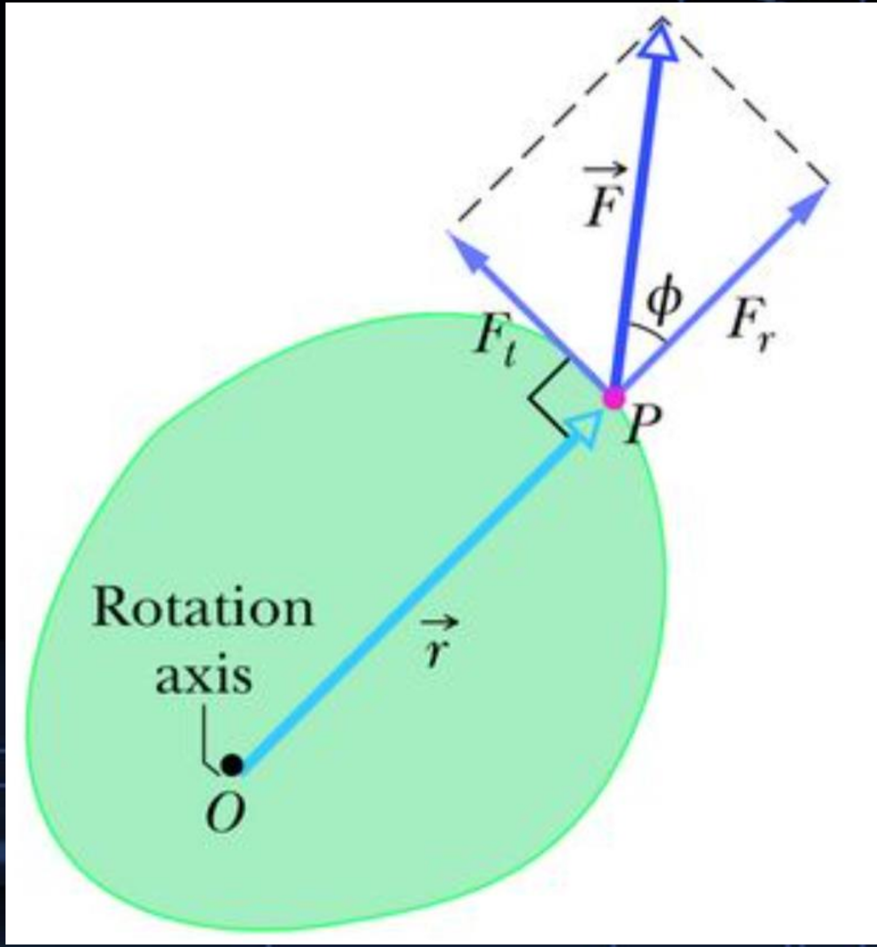
- Consider a cross section of a body that is free to rotate about an axis passing through O and perpendicular to the cross section. A force $\vec{F}$ is applied at point P, whose position relative to O is defined by a position vector $\vec{r}$. The directions of vectors $\vec{F}$ and $\vec{r}$ make an angle ϕ with each other.



Torque

To determine how $\vec{F}$ results in a rotation of the body around the rotation axis, we resolve $\vec{F}$ into two components:

- $\overrightarrow{F_r}$ points along $\vec{r}$. This component does not cause rotation
- $\overrightarrow{F_t}$: the tangential component Ft, is perpendicular to $\vec{r}$ and has magnitude Ft = F sin ϕ. This component does cause rotation



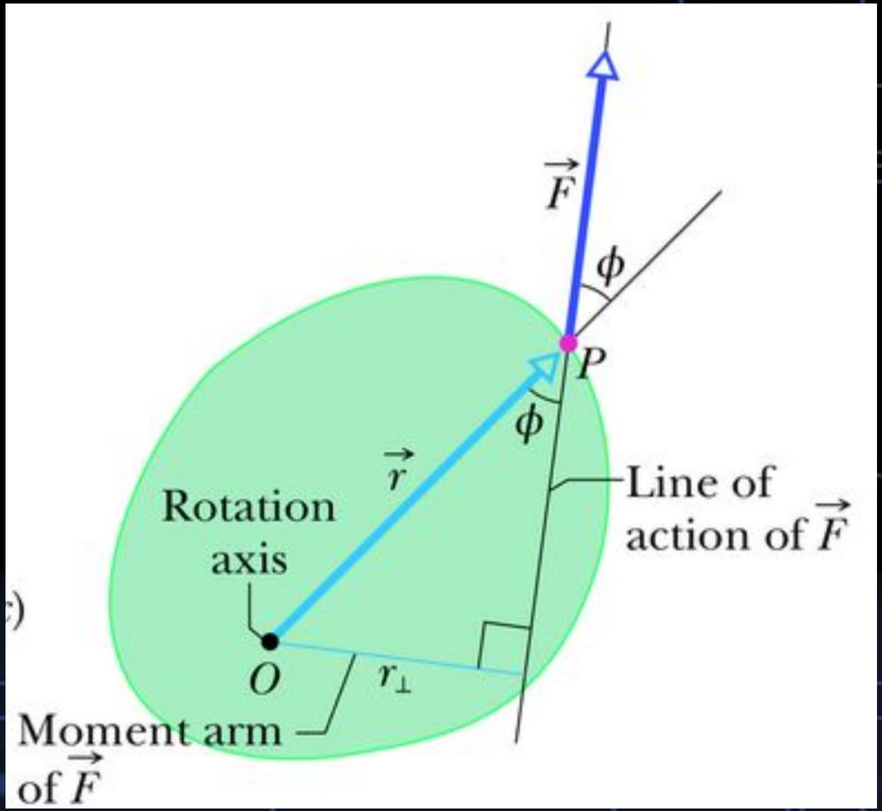
Torque

Then a torque is defined as:

$$\tau = rFsin\phi$$

The two ways to calculate

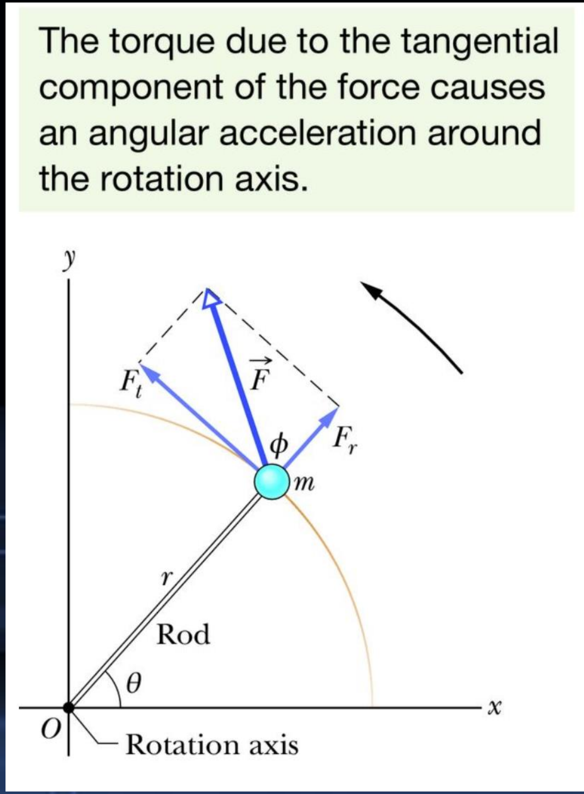
- $\tau = rFsin\phi = rF_{\perp}$
- $\tau = (rsin\phi)F = r_{\perp}F$, where this extended line is called the line of action of and r⊥ is called the moment arm of $\vec{F}$.



Newton’s second law of rotation

- A torque can cause rotation of a rigid body. This can be derived from newton’s second law and one can get:

$$\tau = I\alpha$$



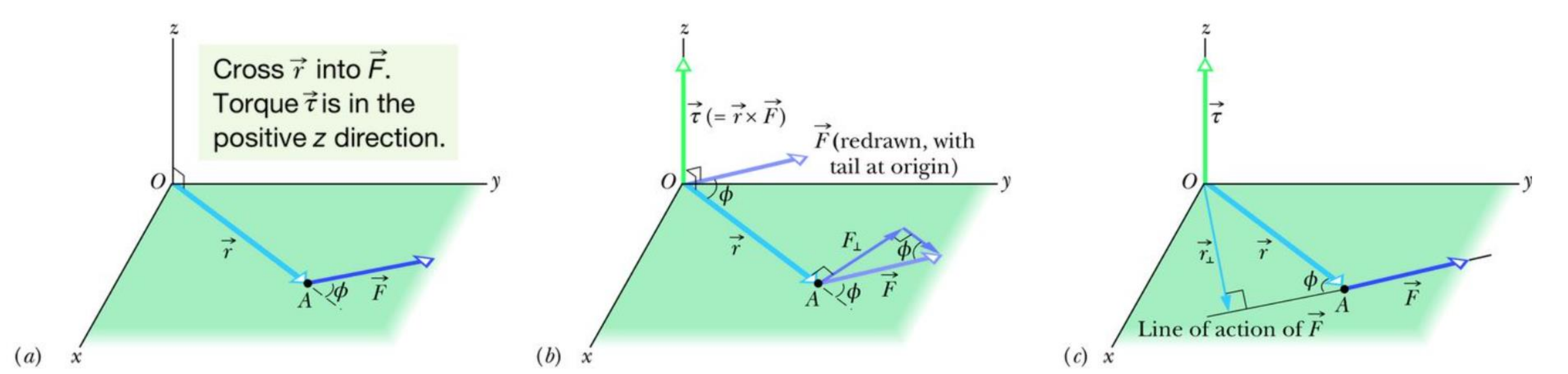
A force acts on the particle and only the tangential component Ft of the force can accelerate the particle along the path:

$$F_t = ma_t = m(\alpha r)$$

$$\tau = F_t r = \alpha r^2 m = I\alpha$$

Define torque as a vector

A single force $\vec{F}$ in that plane acts on the particle, and the particle's position relative to the origin O is given by position vector $\vec{r}$ The torque $\vec{\tau}$ acting on the particle relative to the fixed point O is a vector quantity defined as $\vec{\tau} = \vec{r} \times \vec{F}$



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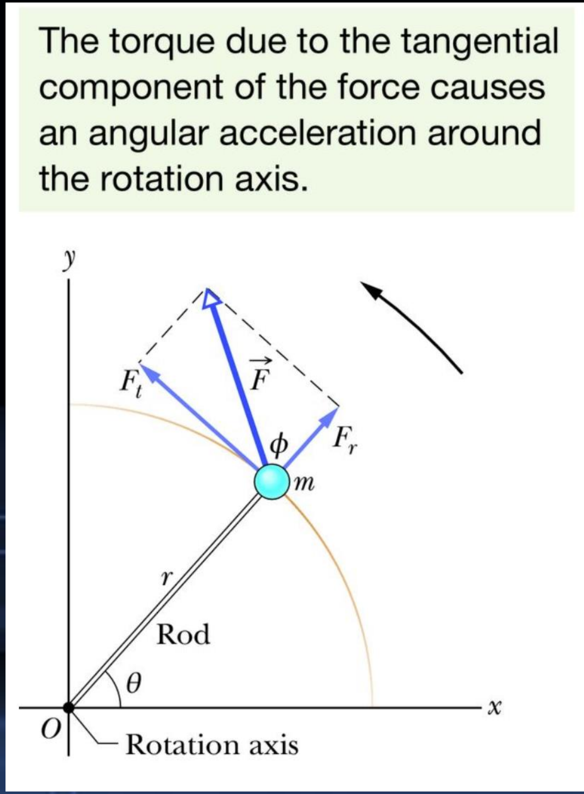

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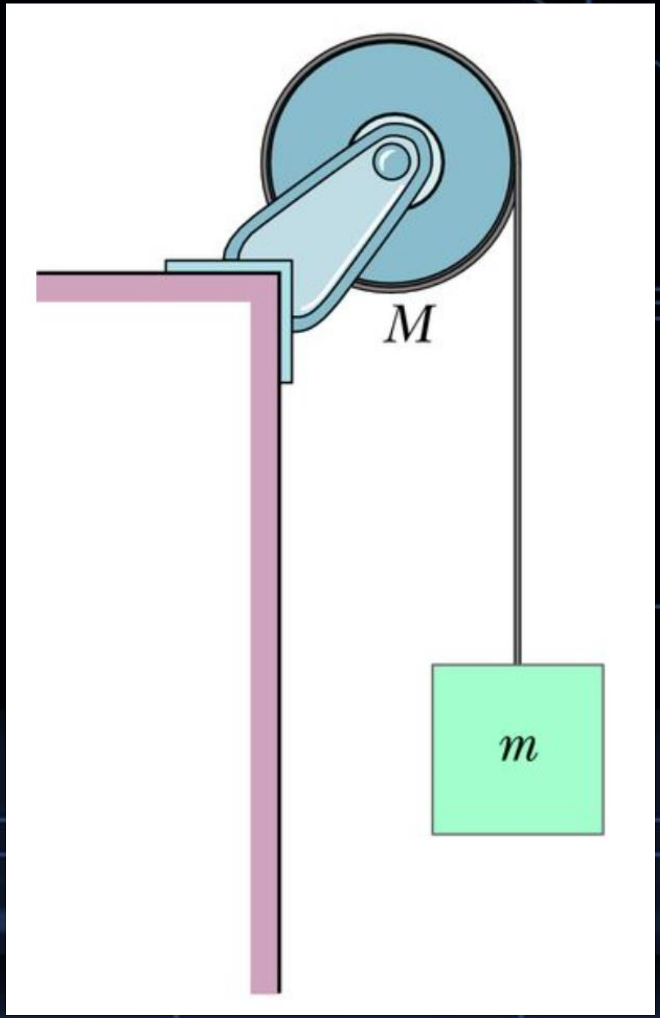
$$\tau = F_t r = \alpha r^2 m = I\alpha$$

Example of Newton's second law of rotation

A uniform disk, with mass M = 2.5 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block with mass m = 1.2 kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord.



Example of Newton's second law of rotation

As shown in the free body diagram, the tenson of cord $\vec{T}$ is exerting a torque on the disk, which has a rotational inertia $I = \frac{1}{2}MR^2$.

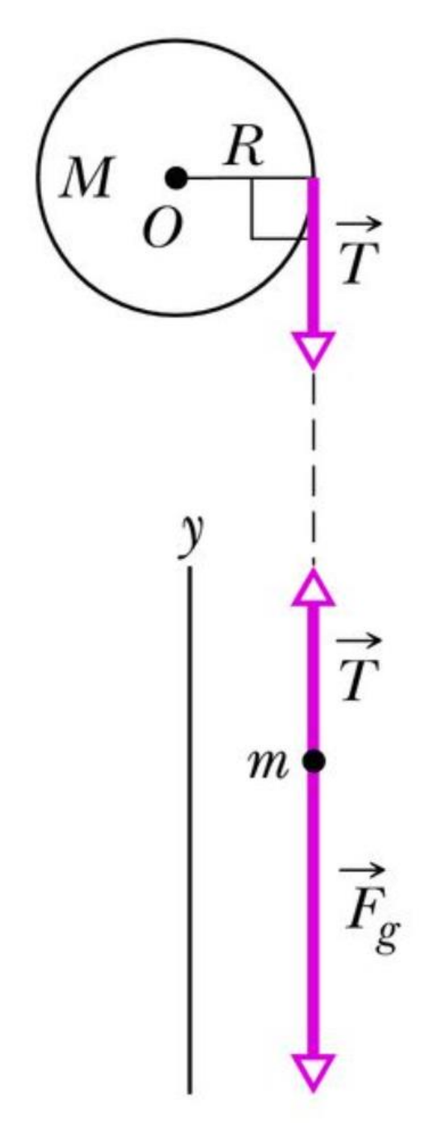
Thus $\tau = \mathrm{TR} = I\alpha = \left(\frac{1}{2}MR^2\right)\left(\frac{a}{R}\right) \Rightarrow \mathrm{T} = \frac{Ma}{2}$

On the mass m, we have $\mathrm{mg} - T = ma$

Thus, we can calculate: $a = \frac{mg}{m+M/2} = 4.8m/s^2$

$$T = Ma/2 = 6.0N$$

$$\alpha = \frac{a}{R} = 24\,rad/s^2$$



Work-Energy theorem for rotation

- As we do in the linear motion, we can also do integral respect to angular displacement on the Newton's second law. Then, we can find:

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta \quad \text{(work, rotation about fixed axis)}$$

$$\Delta K = K_f - K_i = \tfrac{1}{2} I\omega_f^2 - \tfrac{1}{2} I\omega_i^2 = W \quad \text{(work–kinetic energy theorem)}$$

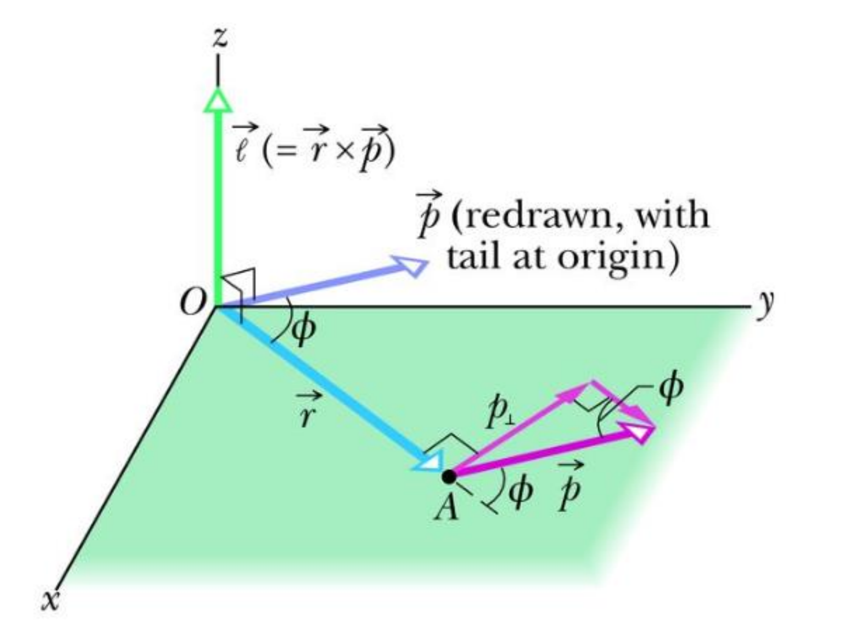
$$P = \frac{dW}{dt} = \tau\omega \quad \text{(power, rotation about fixed axis)}$$

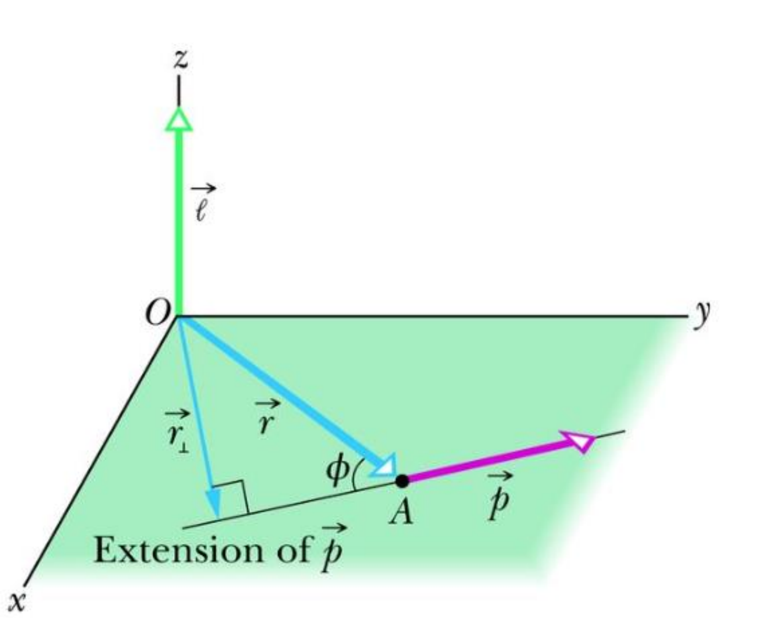
Angular Momentum

- Recall that the concept of linear momentum $\vec{P}$ and the principle of conservation of linear momentum are extremely powerful tools. In rotation, we have same angular counterpart of the conservation principle.

Angular Momentum

A particle of mass m with linear momentum $\vec{p} = m\vec{v}$ as it passes through point A in an xy plane. The angular momentum $\vec{l}$ of this particle with respect to the origin O is a vector quantity defined as $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

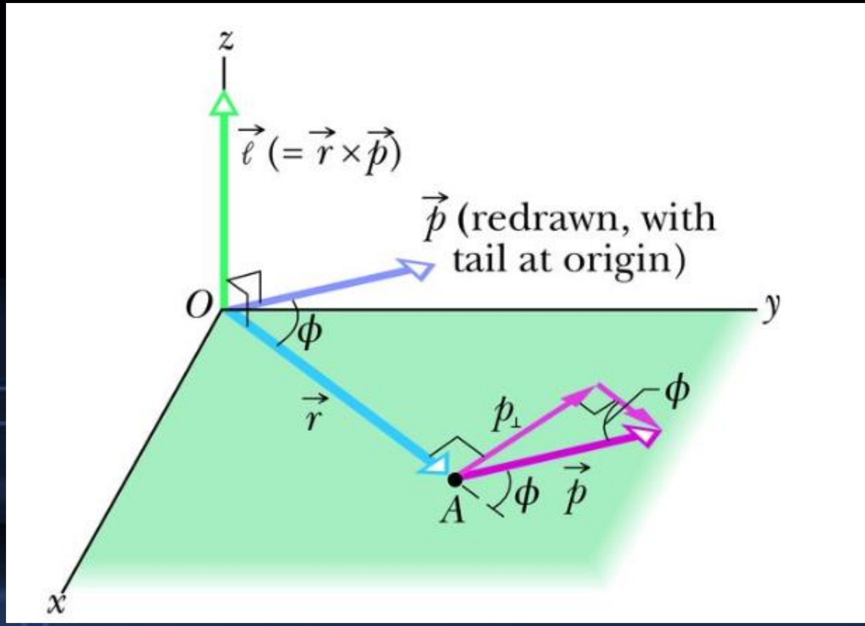


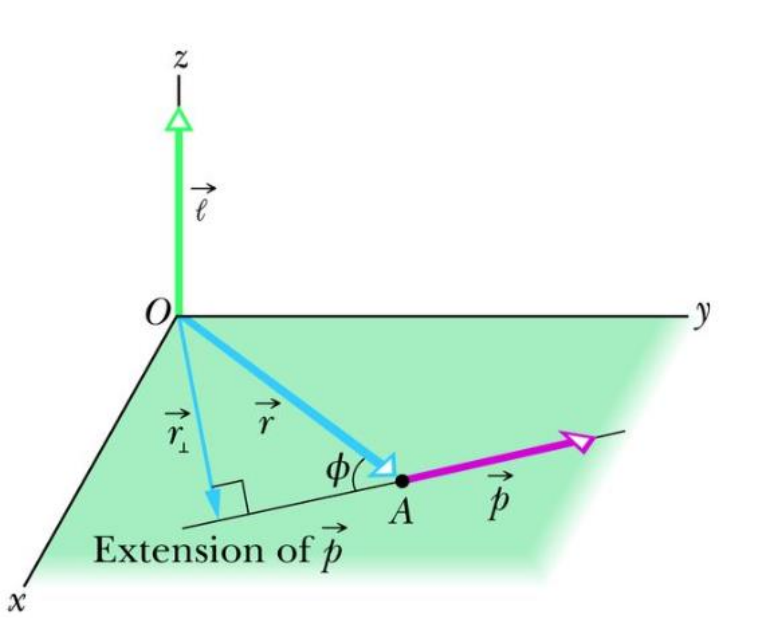


Angular Momentum

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Direction: Right-hand rule for vector products, sweeping the fingers from $\vec{r}$ into $\vec{p}$. The outstretched thumb then shows that the direction of $\vec{l}$.

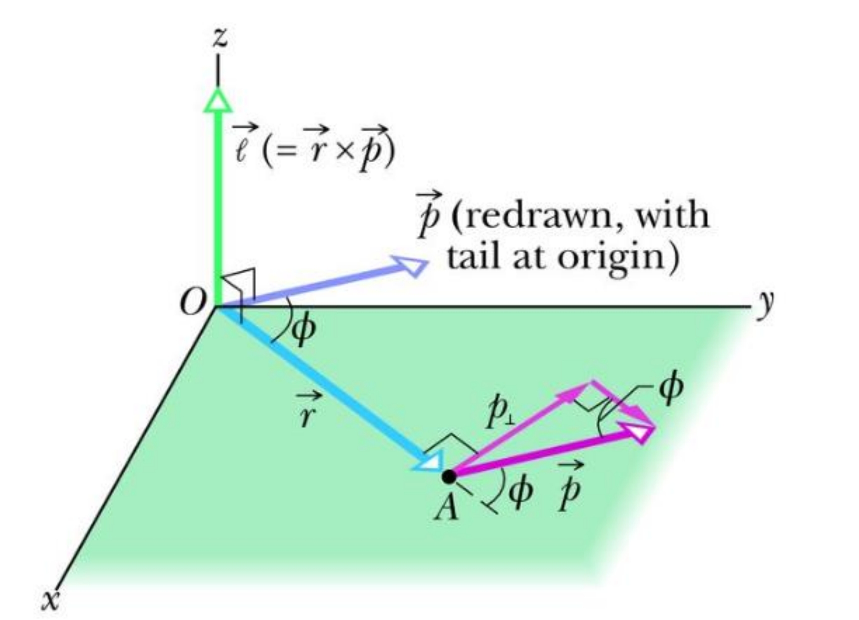


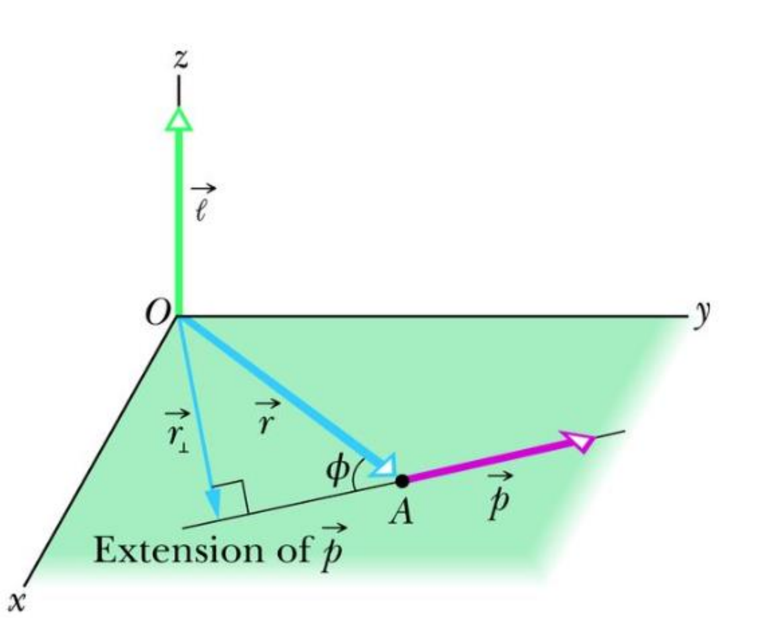


Angular Momentum

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Magnitude: The magnitude of $\vec{l}$ is $l = rmv\ sin\phi$





Newton's second law in angular form

- Newton's second law for a particle can be written in angular form as

$$\overrightarrow{\tau_{NET}} = \frac{d\vec{l}}{dt}$$

where $\overrightarrow{\tau_{NET}}$ is the net torque acting on the particle and $\overrightarrow{\tau_{NET}}$ is the angular momentum of the particle.

Momentum: translational vs. rotational

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\left(=\vec{r}\times\vec{F}\right)$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\left(=\vec{r}\times\vec{p}\right)$
Linear momentum[b]	$\vec{P}\left(=\Sigma\vec{p}_i\right)$	Angular momentum[b]	$\vec{L}\left(=\Sigma\vec{\ell}_i\right)$
Linear momentum[b]	$\vec{P}=M\vec{v}_{\text{com}}$	Angular momentum[c]	$L=I\omega$
Newton's second law[b]	$\vec{F}_{\text{net}}=\frac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{\text{net}}=\frac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}=\text{a constant}$	Conservation law[d]	$\vec{L}=\text{a constant}$

Conservation of angular momentum

The angular momentum $\vec{L}$ of a system remains constant if the net external torque acting on the system is zero:

$$\vec{L} = a\ constant\ (isolated\ system)$$

Or

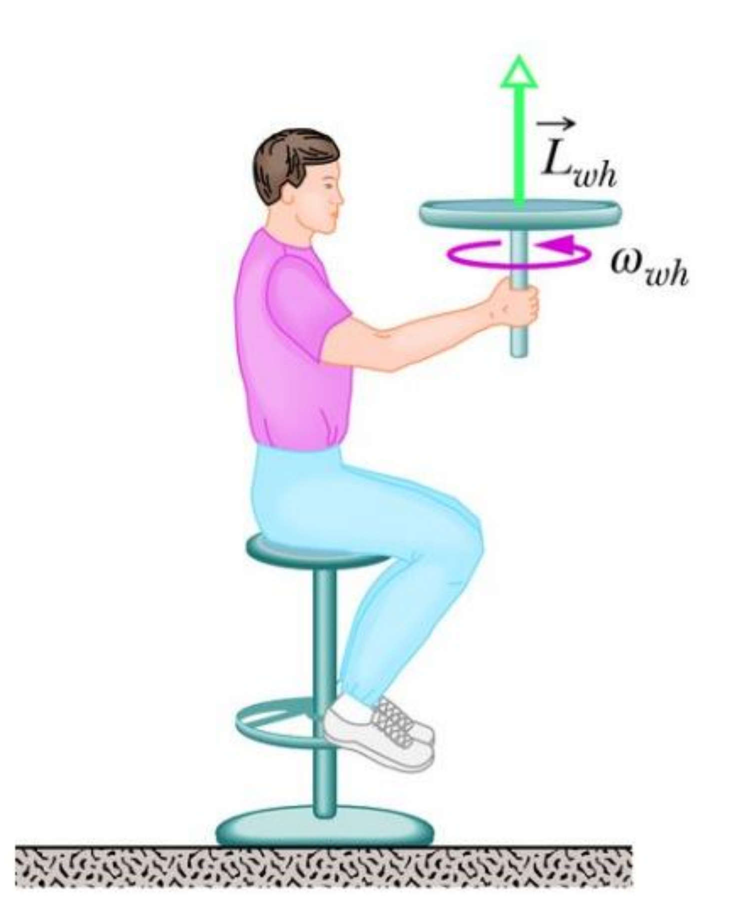
$$\overrightarrow{L_i} = \overrightarrow{L_f}(isolated\ system)$$

This is the law of conservation of angular momentum.

https://gfycat.com/classicdampcavy

Think about it…

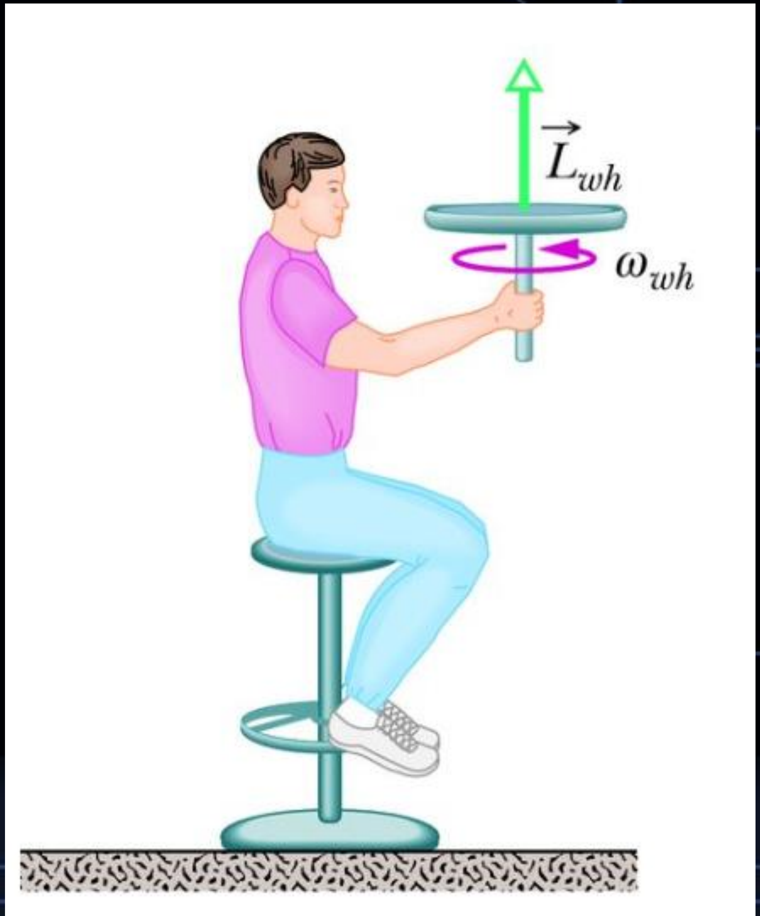
As shown in the figure, a student is sitting on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead. The wheel is rotating as seen from overhead, the rotation is counterclockwise. The student now inverts the wheel. What will happen?



Think about it…

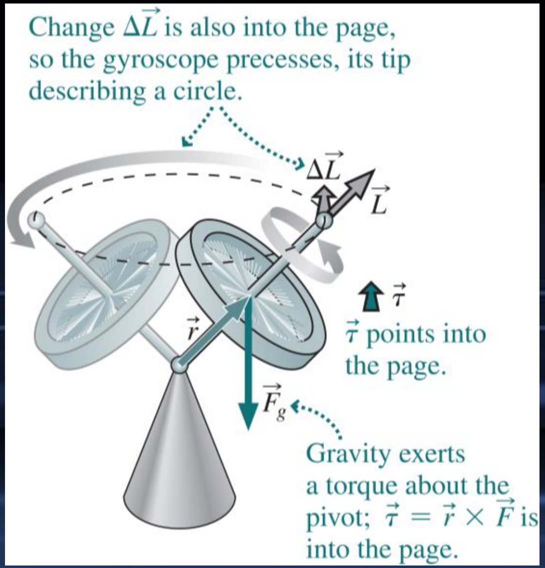
A. The student will still be at rest.
B. The student will rotate clockwise as seen from over head.
C. The student will rotate counterclockwise as seen from over head.
D. Others.

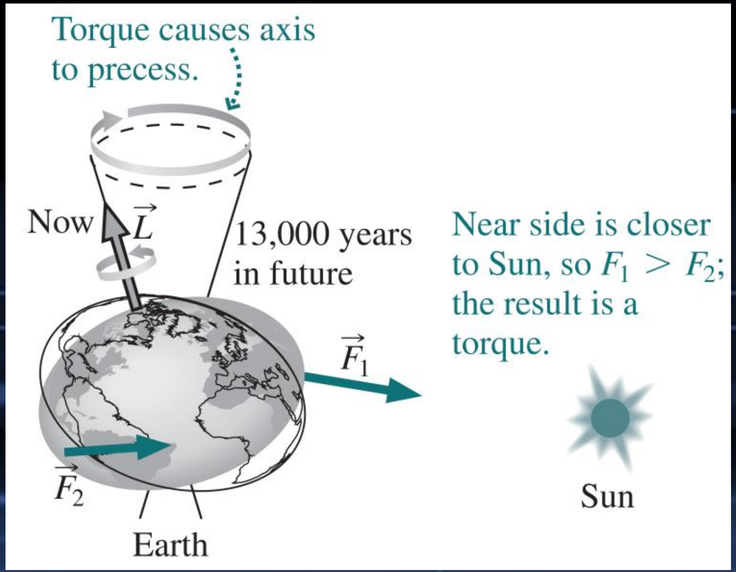
Why?



Gyroscopes and Precession

- Precession occurs when there is a continual change in the rotational axis of a rotating body:
  - Precession can occur in a rotating gyroscope when an external torque acts on it, changing the direction (but not necessarily the magnitude) of its angular momentum vector.
  - As a result, the rotation axis undergoes circular motion:





Summary I

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Summary II

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau} \left(= \vec{r} \times \vec{F}\right)$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell} \left(= \vec{r} \times \vec{p}\right)$
Linear momentum[b]	$\vec{P} \left(= \Sigma \vec{p}_i\right)$	Angular momentum[b]	$\vec{L} \left(= \Sigma \vec{\ell}_i\right)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{\text{com}}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P} = \text{a constant}$	Conservation law[d]	$\vec{L} = \text{a constant}$