Wolfson Ch27 Electromagnetic Induction

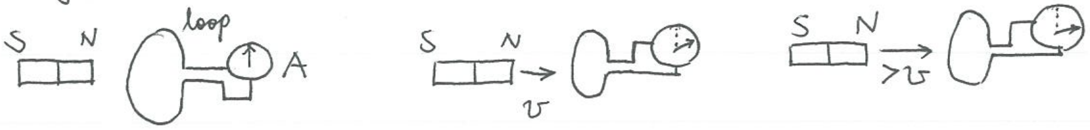
$\vec{B} \to \vec{E}$

(1) Induced currents

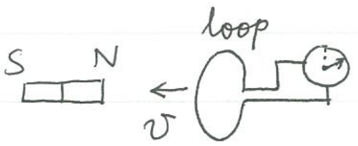
— 1831, Faraday and Henry 所發現

實驗上發現：隨時間變化(時變) $\vec{B}$ 產生感應電流

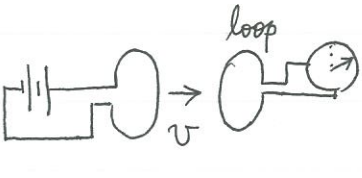
(a) Fig.27.1



(b) Fig.27.2

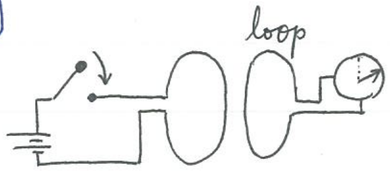


(c) Fig.27.3



a current-carrying circuit replaces the magnet.

(d) Fig.27.4



所有的exp.特徵：通过導線迴路(loop or coil)的B產生時變

——電磁感應(electromagnetic induction), $\vec{B}$ 產生 $\vec{E}$!

(2) Faraday law

How to 定量描述「通过 loop 的 $\vec{B}$」?

Use $\vec{B}$ lines and loop's area:

$\Phi_B = \int \vec{B}\cdot d\vec{A}$ = 通過 loop 的磁通量 (Note：分辨 $\oint \vec{B}\cdot d\vec{A} = 0$)

where $d\vec{A}$ is the surface element of the area enclosed by loop.

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例：均匀$\vec{B}$通过面積為A的loop，則

$$\Phi_B = \vec{B}\cdot\int d\vec{A} = \vec{B}\cdot\vec{A} = B\cdot A\cdot\cos\theta$$

where θ是$\vec{B}$與loop法線的夾角.（$\vec{A}$的方向？）

Faraday law of induction：$\mathcal{E} = -\frac{d}{dt}\Phi_B$

i.e. induced emf = 通过loop的Φ_B的變化率。

Think：電磁感應的實驗中並無像電池般的emf，但在loop中仍產生電流 → 此种induced emf 並非像電池是局部的emf，而是分布在whole loop.

emf → current，表示有induced $\vec{E}$產生、
Then，在沒有loop時，是否仍有induced E存在空間中？
⇒ Yes！此為Faraday law的廣義描述。

"−"号的意義（Lenz law）：induced emf 反抗Φ_B的變化，從而決定感應電流或induced $\vec{E}$的方向.

For uniform $\vec{B}$ (may be time-varying)

$$\begin{aligned} \mathcal{E} &= -\frac{d}{dt}\Phi_B = -\frac{d}{dt}(B\cdot A\cdot\cos\theta)\\ &= -A\cdot\cos\theta\cdot\frac{dB}{dt} \quad (\text{B case, 如Fig.27-1}\sim 27.4)\\ &\quad \left.\begin{array}{l} - B\cdot\cos\theta\cdot\frac{dA}{dt}\ (\text{A case, 如Example 27.4})\\ + B\cdot A\cdot\sin\theta\cdot\frac{d\theta}{dt}\ (\theta\text{ case, 如Generator})\end{array}\right]\text{— motional emf}\end{aligned}$$

motional emf：coil or loop的A or θ（相對於$\vec{B}$）is changing with time → conductor中的charge carriers在$\vec{B}$中運動！

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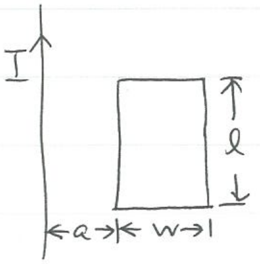
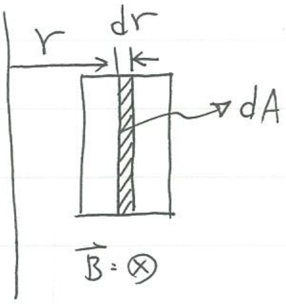
Example 27.2 (B case)



如右圖，通I的长直wire旁的长方形loop，长边l平行於wire，則通过loop的$|\Phi_B| = ?$

→ 长直wire外的 $B = \frac{\mu_0 I}{2\pi r}$ and $\vec{B} \parallel \vec{A}$ or $\vec{B} \parallel -\vec{A}$



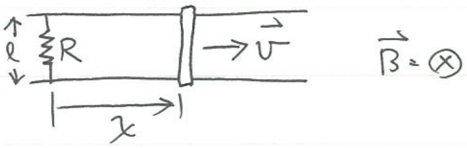
∵ 通过loop的B不是uniform的，
將loop面積作长边方向的切割，
寬度 = dr，长度 l，通过此 $dA = l \cdot dr$
面積的 $B = \frac{\mu_0 I}{2\pi r}$

∴ $|d\Phi_B|$ (通过dA的磁通量)

$$= B(r) \cdot dA = \frac{\mu_0 I}{2\pi r} \cdot l \cdot dr$$

$$\therefore |\Phi_B| = \int |d\Phi_B| = \frac{\mu_0 I l}{2\pi} \int_a^{a+w} \frac{dr}{r} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{a+w}{a}\right)$$

Example 27.4 (A case)



間距l的二平行軌道，一端以電阻R連接，另一端為可無摩擦滑動的金屬bar，全部置於垂直方向上的均勻$\vec{B}$中。When bar以$\vec{v}$向右運動時，電流 = ?

Loop: R + 軌道 + bar 形成 面積 $l \cdot x$ 的loop。
$\vec{B}$ 與loop的 $\vec{A}$ 平行或反平行，∴ $|\Phi_B| = B \cdot A = B \cdot l \cdot x$

When bar 改變位置 → loop的area changes

$$\therefore |\varepsilon| = \left|-\frac{d}{dt}\Phi_B\right| = B \cdot \left|-\frac{dA}{dt}\right| \quad (A\ case)$$

$$= B \cdot l \cdot \frac{dx}{dt} = B \cdot l \cdot v$$

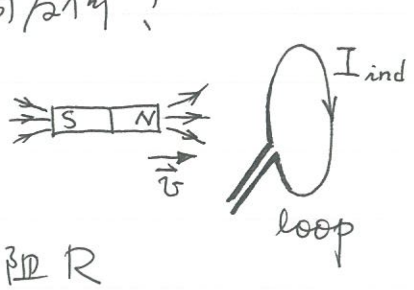
$$= IR$$

$$\therefore I = Blv/R.$$

(3) Induction and induced $\vec{E}$

○ $\mathcal{E} = -\frac{d\Phi_B}{dt}$ 產生的感應電流 I_{ind} 方向為何？

→ from energy 的觀點



在 page 1 中

磁鐵靠近 loop 產生 I_{ind}, loop 有電阻 R

∴ loop 上消耗 energy 的 power 為 $I_{ind}^2 R$.

這些 energy 從何而來？

Obviously, 是從磁鐵運動而來。設磁鐵以等速度 $\vec{v}$ 靠近

→ 推磁鐵前進的外力 $\vec{F}_{ext}$ 在作正功，且磁鐵受到與 $\vec{F}_{ext}$ 反向、大小相等的力排斥（∵ 磁鐵以 $\vec{v}$ 前進，∴ $\vec{F}_{net} = 0$）

→ 此排斥力顯然是 I_{ind} 產生的 $\vec{B}_{ind}$（向左）對磁鐵的斥力

→ 決定了 I_{ind} 的方向.

同理在磁鐵以 $\vec{v}$ 遠離時，磁鐵受到 I_{ind} 產生的 $\vec{B}_{ind}$ 的吸力，
→ 外力作正功。

Lenz law ：（能量守恆 + 電磁感應）

loop 中的 I_{ind} 產生 $\vec{B}_{ind}$ 與外磁場 $\vec{B}$ 的 $\frac{d\Phi_B}{dt}$ 相反 or $\vec{B}_{ind}$ 欲維持通過 loop 的總磁通量 $(\Phi_B + \Phi_{B_{ind}})$ 不變.

○ Motional emf (A or D case)

Conductor 在 $\vec{B}$ 中運動 ⇒ conductor 內的 charge carriers 在 $\vec{B}$ 中運動，受磁力 $\vec{F} = q\vec{v} \times \vec{B}$ 的作用形成電流，亦是感應電流.

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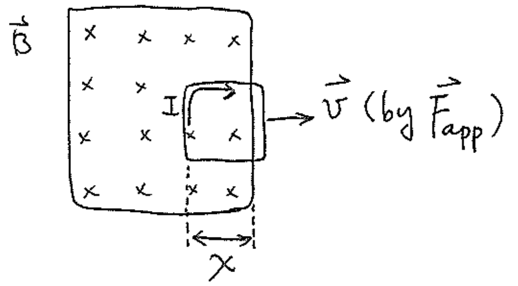
Wolfson ch27

(also see Example 27.4 in page 3)



邊長l, 電阻為R的正方形loop, 被外力$\vec{F}_{app}$以$\vec{v}$等速拉離$\vec{B}$.
(A case)

→ $\vec{B}$作用在loop內carriers形成順時針電流 I (see textbook), 此電流方向和Lenz law的感應電流同方向。

I = ?, I流經loop消耗的energy從何而來?

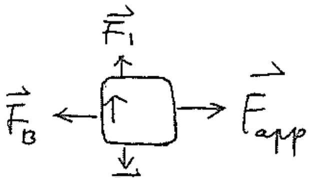
$$\varepsilon = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}(B\cdot l\cdot x) = B\cdot l\cdot\left(-\frac{dx}{dt}\right) = Blv = IR$$

$$\therefore I = \frac{Blv}{R}$$

→ 在loop上消耗的power $P_R = I^2R = \frac{(Blv)^2}{R}$.

唯一在作功的對象:外力$\vec{F}_{app}$ and $P_{app} = v\cdot F_{app}$

Where $F_{app} = F_B = IlB$

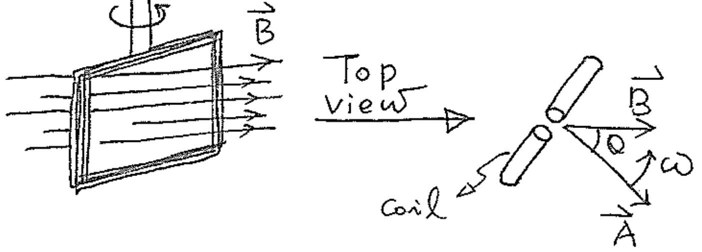


$\vec{F}_1$和$\vec{F}_2$抵消

$$\rightarrow P_{app} = v\cdot IlB = (Blv)^2/R$$
$$= P_R$$

Motional emf 的重要應用: Generator (θ case)

N-turn coil in uniform $\vec{B}$



$$\Phi_B = N\vec{B}\cdot\vec{A} = N\cdot B\cdot A\cos\theta = NBA\cos\omega t$$ (coil 以ω 轉動)

則 $\varepsilon = -\frac{d}{dt}\Phi_B = NBA\omega\sin\omega t = \varepsilon_0\sin\omega t$ (交流電)

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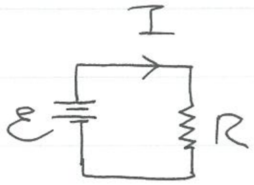
o Induced $\vec{E}$



emf:

右圖的 $P = \frac{\Delta W}{\Delta t} = I\mathcal{E} = \frac{\Delta Q}{\Delta t}\cdot\mathcal{E}$

∴ $\mathcal{E} = \frac{\Delta W}{\Delta Q}$

= 將單位 charge 繞行 closed loop 一圈 (形式 I)

所需作的功 ΔW.

∴ $\mathcal{E} = \frac{\Delta W}{\Delta Q} = \frac{1}{\Delta Q}\oint \vec{F}\cdot d\vec{r}$ where $\vec{F}$ = 施加在 ΔQ 上的作用力。

When $(\vec{E}+\vec{B})$ 時, ΔQ 所受之力 $\vec{F} = \Delta Q(\vec{E}+\vec{v}\times\vec{B})$

where $\vec{v}$ is the velocity of ΔQ

∴ $\mathcal{E} = \oint(\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{r} = \oint\vec{E}\cdot d\vec{r} + \oint(\vec{v}\times\vec{B})\cdot d\vec{r}$

後者 $\oint(\vec{v}\times\vec{B})\cdot d\vec{r}$ = motional emf, 因 ΔQ 在 $\vec{B}$ 中運動 所造成, 即 A or θ case.

when $\vec{v}=0$ 即 conductor 在 $\vec{B}$ 中的 loop $\vec{A}$ 及 θ 不變時,

$$\mathcal{E} = \oint\vec{E}\cdot d\vec{r} = -\frac{d}{dt}\Phi_B = -\left(\frac{dB}{dt}\right)\cdot A\cdot\cos\theta$$

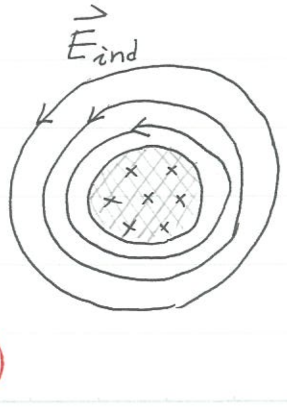
即 B case, Fig. 27.1~Fig. 27.4 皆屬之.

以上處的 $\vec{E}$ = induced electric field $\vec{E}_{ind}$

為 a special field: not a 保守場

(cf: 由 charges 所生之 $\vec{E}$ 為保守場, 特徵: $\oint\vec{E}\cdot d\vec{r}=0$,
但 $\oint\vec{E}_{ind}\cdot d\vec{r} = -\frac{d}{dt}\Phi_B \neq 0$, ∴ $\vec{E}_{ind}$ 為非保守場)

$\vec{E}_{ind}$ lines: 封閉的 loops
方向由 Lenz law 決定。
在有 $\vec{E}_{ind}$ 處置入 loop 即
產生 induced I, 方向 = $\vec{E}_{ind}$。



$\vec{B}$: ⊗ and $\frac{dB}{dt} > 0$

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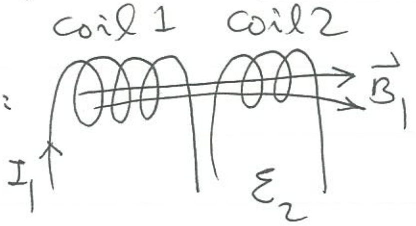
電磁感應有廣泛的應用
→ eddy current
→ transformer、generator
→ diamagnetism (Fig. 27.32)

(4) Inductance and RL circuits

o 互感 (mutual inductance) and 自感 (self-inductance)

互感：



由coil 1產生的$\vec{B}_1$ lines在coil 2形成的Φ_{B_1}，產生

$\mathcal{E}_2 = -\frac{d}{dt}\Phi_{B_1}$

→ Max $\mathcal{E}_2$ occurs when all $\vec{B}_1$ lines go through coil 2.

互感是 transformer (ch28)的運作原理。

自感：只要有closed loop便能形成induced電流。

⇒ 產生$\vec{B}$的coil本身就會產生induced電流，此induced電流抵抗$\vec{B}$的變化 (completely B case).

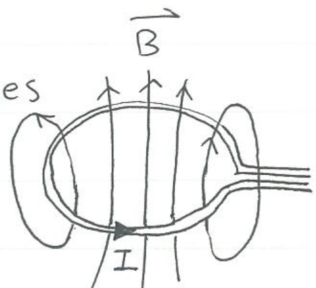
如右图：I產生$\vec{B}$，$\vec{B}$ changes ↔ I changes

∴產生的I_{ind}反抗I的change

when B↑ ↔ I↑，$I_{ind.}$與I反向；

when B↓ ↔ I↓，I_{ind}與I同向。

$I_{ind.}$與通过coil or loop的Φ_B有關.



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Inductor (電感, 符号 —ooo—, 用 L 表示)

就是 solenoid or N-turn 的 coil.

when 有 I 流通的 solenoid, 通过 1 圈的磁通量 $= B\cdot A = \mu_0 n I \cdot A$

∴ 穿過長度 l 的 solenoid 的總磁通量 $\Phi_B = (n\cdot l)\cdot(B\cdot A) = \mu_0 n^2 (A\cdot l)\cdot I$

, where A = selenoid 的 cross section area.

Define $\Phi_B \equiv L\cdot I$

$$\Rightarrow L = \Phi_B / I \text{ (self-inductance)}$$

$$[L] = T\cdot m^2/A \equiv \text{hery (用 H 表示)}$$

For solenoid, $L = \mu_0 n^2 A\cdot l$ (為螺旋管的几何參數)

∴ for an inductor, 其 induced emf

$$\varepsilon_L = -\frac{d}{dt}\Phi_B = -L\frac{dI}{dt}$$

ε_L 和 $\frac{dI}{dt}$ 反号, ∴ ε_L is called a back emf, 產生的 I_{ind} 和 I 變化方向相反.

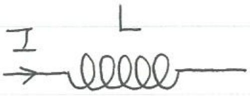
→ I_{ind} 阻緩 I 的變化

Start or stop current suddenly

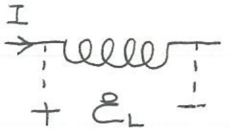
→ $\frac{dI}{dt}$ goes very large and a very large back emf appears.

○ RL circuit

跨越L的電位變化：

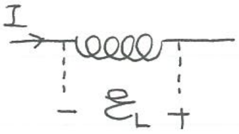


(i) $\frac{dI}{dt}>0$, 則



i.e. 沿I方向經过L的電位變化為 $\mathcal{E}_L=-L\frac{dI}{dt}$ $\left(<0, \because \frac{dI}{dt}>0\right)$

(ii) $\frac{dI}{dt}<0$, 則

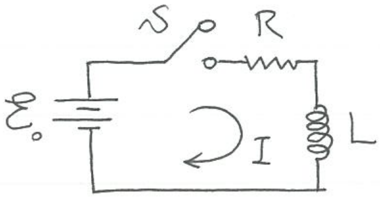


沿I方向經过L的電位變化為 $\mathcal{E}_L=-L\frac{dI}{dt}$ $\left(>0, \because \frac{dI}{dt}<0\right)$

→ Loop rule: 沿I方向經过L的電位變化為$\left(-L\frac{dI}{dt}\right)$。

跨越C的voltage無法瞬間達極值, so does L：流过L的電流亦因back emf而無法瞬間達極值、

(a) 電流上升



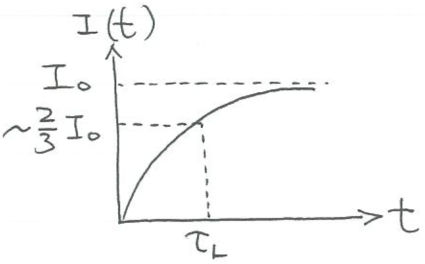
起始條件：t=0, S closes.

loop rule: $\mathcal{E}_0-IR-L\frac{dI}{dt}=0$, I(t)=?

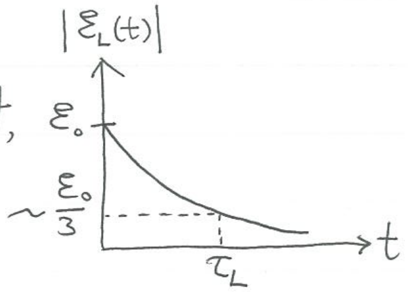
變數分離+積分：$\frac{dI}{IR-\mathcal{E}_0}=-\frac{dt}{L}$

∴, $I(t)=\frac{\mathcal{E}_0}{R}\left(1-e^{-Rt/L}\right)\equiv I_0\left(1-e^{-Rt/L}\right)$

and $\mathcal{E}_L=-L\frac{dI}{dt}=IR-\mathcal{E}_0=-\mathcal{E}_0e^{-Rt/L}$



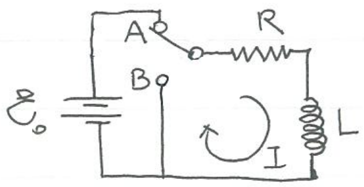
$\tau_L=\frac{L}{R}$ = inductive time constant,



⇒ L在t=0是斷路

在t→∞是通路(無電阻的wire)

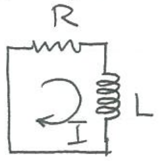
(b) L中的電流下降



起始條件：t<0, S to A

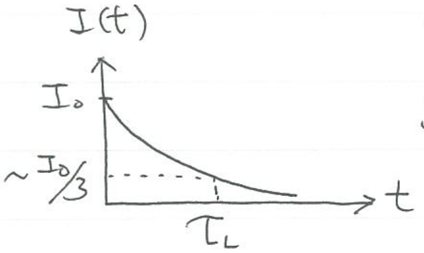
at t=0, S to B，∴ L中的 I，因R的消耗energy而decay.

at t=0, $I=I_0=\max$.



$$\Rightarrow -L\frac{dI}{dt}-IR=0$$

$$\therefore I(t)=I_0e^{-Rt/L}=\frac{\mathcal{E}_0}{R}e^{-Rt/L}$$



∴ at t=0, L是像R=0的wire

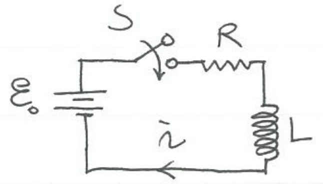
as t→∞, L是斷路.

(5) $\vec{B}$中的energy

~ $\vec{E}$, $\vec{B}$的建立代表energy的storage, 有$\vec{B}$即有energy.

RC circuit ~ RL circuit.

Check:



在t=0, S closes，∴ L中的電流由 0→max.

When 電路中的電流為 i 時

$$\mathcal{E}_0=iR+L\frac{di}{dt},\ (\times i)\Rightarrow \underbrace{i\mathcal{E}_0}_{\mathcal{E}_0\text{所供應的power}}=\underbrace{i^2R}_{R\text{所消耗的power}}+\underbrace{Li\frac{di}{dt}}_{\text{power stored in }L}$$

設 U_B = store 在L的energy, 則

$$\frac{dU_B}{dt}=\text{power stored in }L=Li\frac{di}{dt}\Rightarrow dU_B=L\,i\,di$$

∴ 當L的電流：0→I 時, energy stored in L is

$$U_B=\int dU_B=\int_0^I L\cdot i\cdot di+C\quad (C=0,\ \because i=0,\ U=0)$$

$$=\frac{1}{2}LI^2\qquad \left(\text{compared with } U_E=\frac{1}{2C}Q^2\right)$$

For a solenoid (or inductor) $L=\mu_0 n^2 l A$

$=\mu_0 n^2 \cdot V$ (V= solenoid 体積) —①

and L 中的 $B=\mu_0 n I$, ∴ $I=\frac{B}{\mu_0 n}$ —②

①、② 代入 $U_B=\frac{1}{2}LI^2$

$$\Rightarrow U_B=\frac{B^2}{2\mu_0}\cdot V$$

Define energy density of B

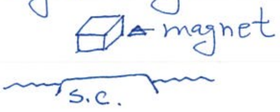
$$u_B=\frac{U_B}{V}=\frac{1}{2\mu_0}B^2 \quad (\text{compared with } u_E=\frac{1}{2}\epsilon_0 E^2)$$

只要有 $\vec{B}$, 就有 energy, 不局限是 solenoid or inductor.

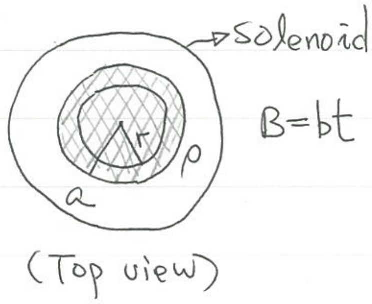
⇒ 反磁性 (diamagnetism)

→ superconductors are perfectly diamagnetic

→ Meissner effect



Problem 69



(Top view)

$B=bt$

(a) loop of radius r

$|\Phi_B|=\pi r^2 \cdot bt$

∴ $|\varepsilon|=\left|-\frac{d}{dt}\Phi_B\right|=\pi r^2 \cdot b=\oint \vec{E}\cdot d\vec{r}=E(r)\cdot 2\pi r$

∴ $E(r)=\frac{b}{2}r=\rho\cdot J(r) \Rightarrow J(r)=\frac{b}{2\rho}r$

(b) $dP=J(r)\cdot \underline{dA(r)}\cdot|\varepsilon(r)|=\frac{b}{2\rho}r\cdot\underline{h\cdot dr}\cdot\pi r^2 b=\frac{\pi b^2 h}{2\rho}r^3 dr$

$$P=\int_0^a dP=\frac{\pi h a^4 b^2}{8\rho}=\frac{a^2b^2}{8\rho}V \quad (\text{dimension is ok})$$

where V is the volume of disk.

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