Wolfson ch 13 Oscillatory motion

Oscillatory motion occurs throughout the physical world.
It is universal because systems in stable equilibrium naturally tend to return toward equilibrium no matter how they are displaced.
不但是 universal，數學描述方式更是相同。

1、振盪與簡諧運動(SHM)

○ 三個重要量用於描述振盪(oscillation)：

(i) 振幅(amplitude)：距離平衡點的 max. 位移.

(ii) 週期(period, T) = 頻率$^{-1}$：週而復始一次的時間，∴ [T] = s

頻率(frequency, f)：1 sec 內，週而復始的次數，

∴ $[f] = s^{-1} \equiv Hz$.

(iii) 相位常數(phase constant，以 ϕ 表示)：起始條件.

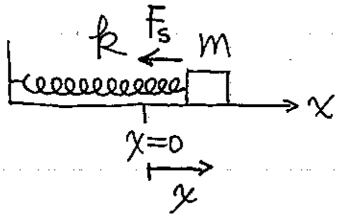
振幅與頻率尚無法完全詳細地描述振盪運動，如 Fig. 13.2，有兩同 A 及 f 的 oscillation 可以有不同的函數形式，此函數形式反應的是回復力(restoring force)的差異.

大部份的物理系統都有相同的回復力形式，即 ideal spring 的彈力形式。

○ Simple harmonic motion（簡諧運動，以 SHM 表示）

以 ideal spring 作用力為 restoring force 的振盪運動：SHM.

為大部分真實系統的近似運動，對小振幅振盪尤其適用.



: mass-spring system with no friction

作用在 m 的力：$F_s = -kx$ = restoring force
回復力
(回復到平衡點)

雜

m受外力作用偏離平衡位置 $x=0$，外力停止作用後，只剩 spring 的回復力 F_s 作用在 m，m 即進行 SHM。

$\rightarrow$ m 的位置函數 $x(t)=?$ $(x\rightarrow v\rightarrow a\rightarrow K\rightarrow U\rightarrow E)$

From Newton II: $F_s = ma = m\frac{d^2}{dt^2}x(t) = -kx(t)$

$$\therefore m\frac{d^2x}{dt^2}+kx=0$$

一元二次微分eq.的通解為 $x(t)=A\cos(\omega t+\phi)$

A 為振幅，$\omega=?$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 \underline{A\cos(\omega t+\phi)} = -\omega^2 x = -\frac{k}{m}x$$

$\therefore \omega=\sqrt{\frac{k}{m}}$ = angular frequency

又 from $x(t)=A\cos(\omega t+\phi)$，cos 的週期為 2π，i.e.

在 $t=T=$週期時 $\omega T=2\pi \Rightarrow \omega=\frac{2\pi}{T}$

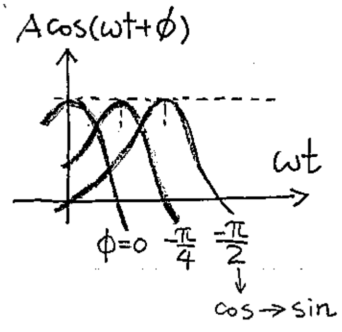
$\therefore [\omega]=\text{rad}\cdot s^{-1}=2\pi f$

Note: A 與 f 無關 in SHM，$\because$ restoring force $\propto x$。

If $F_s=-kx$ 不能描述振盪的回復力，則 f does depend on A, especially, when the displacement x gets too big.

$\therefore$ SHM 僅適用在小振幅的振盪運動。

ϕ: phase constant



$\phi=0$: take max. x at $t=0$
(在 $t=0$，將 m 拉到 A)

$\phi>0$: shift cosine to 左

$\phi<0$: shift cosino to 右

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○ SHM 的 v and a

Take $\phi=0$, i.e. $t=0$, $x(t)=A$

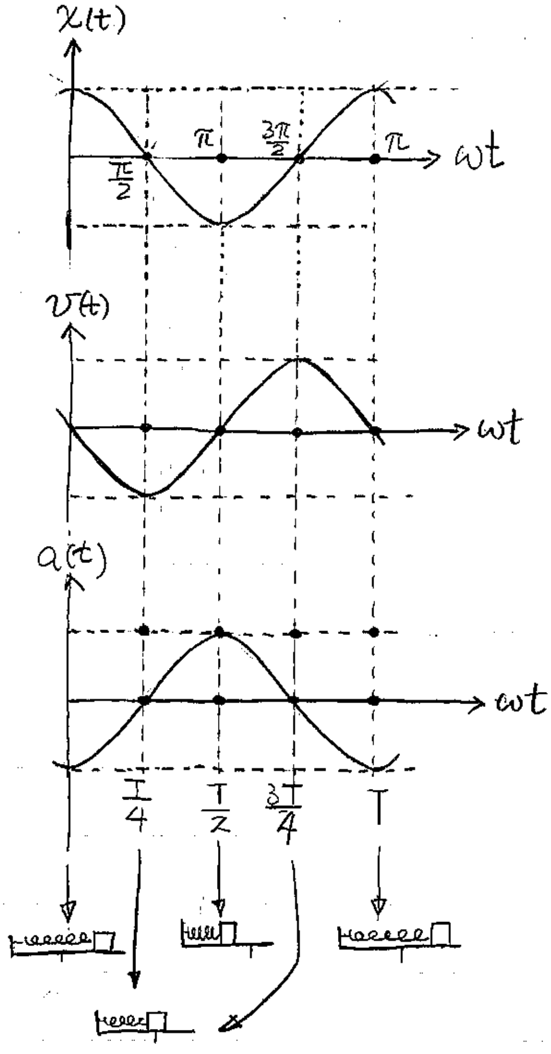
$\therefore x(t)=A\cos\omega t$

$$v(t)=\frac{dx(t)}{dt}=-\omega A\sin\omega t=\omega A\cos(\omega t+\frac{\pi}{2})$$

$$a(t)=\frac{d^2x(t)}{dt^2}=-\omega^2A\cos\omega t=\omega^2A\cos(\omega t+\pi)$$

$\Rightarrow v(t)$ leads (領先) $x(t)$ 達 $\frac{\pi}{2}$ (i.e. $\frac{T}{4}$)

而 $a(t)$ leads $x(t)$ π, 即 $\frac{T}{2}$ 或可以說 lags (落後) $x(t)$ $\frac{T}{2}$。



turning point occurs at

$\omega t=n\pi$, i.e. $t=\frac{n}{2}T$

平衡點(自然長度)的經過

時間點 $\omega t=(n+\frac{1}{2})\pi$

$n\in\mathbb{Z}$, i.e. $t=\frac{T}{4},\frac{3}{4}T,\frac{5}{4}T,\cdots$

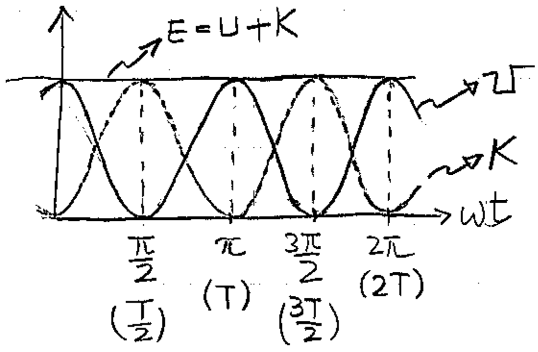
在 turning point: $v=0$, a 達 max.

在平衡點: v 達 max, $a=0$.

2. Energy in SHM

$$E = K + U \quad \begin{cases} K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2A^2\sin^2\omega t \quad (\text{let } \phi = 0) \\ U = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2)x^2 = \frac{1}{2}m\omega^2A^2\cos^2\omega t \end{cases}$$

$$\therefore E = \frac{1}{2}m\omega^2A^2 = \text{constant} = \frac{1}{2}kA^2 \left(\text{by } \omega = \sqrt{\frac{k}{m}}\right)$$



$\Rightarrow$ 從能量守恆導出描述運動的微分 eq. (problem 62)

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$, wher $x = x(t)$ and $v = v(t)$.

(i) $\frac{dE}{dt} = 0 = mv \cdot \frac{dv}{dt} + kx \cdot \frac{dx}{dt}$

$= mv\frac{d^2x}{dt^2} + kx \cdot v$

$\therefore m\frac{d^2x}{dt^2} + kx = 0$

(ii) $\frac{dE}{dx} = 0 = \frac{dE}{dt}\frac{dt}{dx} = \frac{dE}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} = v^{-1} \cdot \frac{dE}{dt}$

$= m\frac{d^2x}{dt^2} + kx$

$\therefore$ $\left.\begin{array}{l}\text{Newton II} \\ \text{能量守恆}\end{array}\right\} \Rightarrow$ 運動微分 eq.

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3. SHM的應用

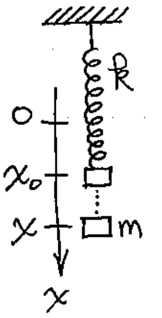
(i) 垂直的 spring-mass system

(ii) Simple pendulum (單擺)

(iii) torsional oscillator (扭力擺)

(iv) physical pendulum

(i) Energy 方法



$x=0=$ 自然長度

$x=x_0$：有m時的平衡位置.

$\therefore mg=kx_0$

$$E=K+U=\text{constant}$$
$$=\frac{1}{2}mv^2+\frac{1}{2}kx^2-mgx$$

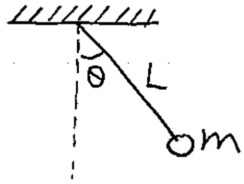
$$\therefore \frac{dE}{dt}=0=mv\frac{dv}{dt}+kx\frac{dx}{dt}-mg\frac{dx}{dt}$$

$$\Rightarrow m\frac{dv}{dt}+kx-mg=0 \quad \because \frac{dx}{dt}=v.$$

$$\therefore m\frac{d^2}{dt^2}x+k(x-x_0)=0, \text{ let } x'=x-x_0, \text{ 則 } \frac{d^2x}{dt^2}=\frac{d^2x'}{dt^2}$$

$$\therefore m\frac{d^2x'}{dt^2}+kx'=0 \text{ — SHM.}$$

(ii) 單擺



① Newton II (DIY)

② 能量方式

$$E=K+U=\text{constant}$$
$$=\frac{1}{2}mv^2+mgL(1-\cos\theta) \quad (U=0 \text{ at the lowest point})$$
$$=\frac{1}{2}m(L\Omega)^2+mgL(1-\cos\theta), \text{ where } \Omega=\text{angular speed}=\frac{d\theta}{dt}$$
$$=\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2+mgL(1-\cos\theta)$$

$$\frac{dE}{dt}=0=mL^2\left(\frac{d\theta}{dt}\right)\cdot\frac{d^2\theta}{dt^2}+mgL\sin\theta\cdot\frac{d\theta}{dt}$$

$$\therefore L\frac{d^2\theta}{dt^2}+g\sin\theta=0$$

for 小角度擺動, i.e. $\theta\ll 1$, then $\sin\theta\simeq\theta$

$$\therefore L\frac{d^2\theta}{dt^2}+g\theta=0 \text{ (運動微分方程) } \left(\sim m\frac{d^2x}{dt^2}+kx=0\right)$$

$$\therefore \omega=\sqrt{\frac{g}{L}} \quad \text{and} \quad T=2\pi\sqrt{\frac{L}{g}} \text{ (和 m 及擺幅無關)}$$

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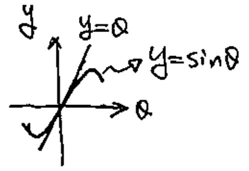
小角度到底要多小？ i.e. θ 要多小，$\sin\theta \approx \theta$ ？

Check：

θ (度)	$57.3°$	$28.65°$	$11.46°$	$5.73°$	$2.87°$
θ (rad.)	1	0.5	0.2	0.1	0.05
$\sin\theta$	0.84	0.48	0.199	0.0998	0.05
$\frac{\lvert\theta-\sin\theta\rvert}{\theta}$	16%	4%	0.5%	0.2%	0

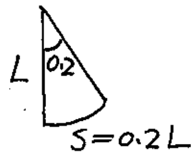


$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$\Rightarrow$ $\theta \le 0.2$ rad，$\sin\theta \sim \theta$，i.e. $s \le 0.2L = 20\% L$

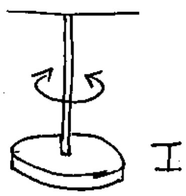
∴ 相同擺幅之下 (i.e. s 相同)

L 愈大，擺鐘愈準。



(iii) Torsional oscillation

為轉動的 Hooke's law



$E = K + U$

$= \frac{1}{2} I \Omega^2 + \frac{1}{2} \kappa \theta^2$, where κ = torsional constant

θ = angular displacement

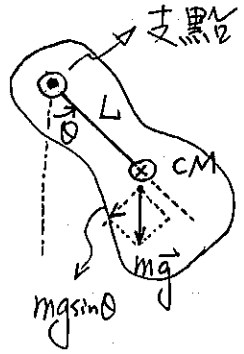
and $\frac{d\theta}{dt} = \Omega$

∴ E 的形式 $\sim \frac{1}{2} m v^2 + \frac{1}{2} k x^2$ in ideal spring system

∴ $\omega = \sqrt{\frac{\kappa}{I}}$

Note : $U(\theta) \Rightarrow -\frac{dU(\theta)}{d\theta} = \tau(\theta) = -\kappa\theta$, ∴ $U(\theta) = \frac{1}{2}\kappa\theta^2 + \text{constant}$.

(iv) physical pendulum



支點不在CM的不規則物体

$\tau = I\alpha$ and

$\vec{\tau} = \vec{L} \times (m\vec{g})$

$\vec{\tau}$ 的方向⊗與 θ 的方向相反

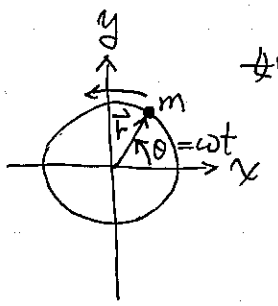
∴ $\tau = -mgL\sin\theta \approx -mgL\theta$ (for small θ)

$= I\alpha = I\frac{d^2\theta}{d\theta^2}$

∴ $I\frac{d^2\theta}{dt^2} + mgL\theta = 0 \Rightarrow \omega = \sqrt{\frac{mgL}{I}}$, $T = 2\pi\sqrt{\frac{I}{mgL}}$

測量T可得I，再由平行軸定理可得 I_{CM}.

4. 等速率圓周運動(UCM) vs. SHM



如左圖的UCM，設m的 $\vec{r}$ 與 x-axis 的夾角為 θ，

若 m 以 ω 作 ccw 的 UCM 則 $\theta = \omega t$

⇒ $\vec{r}$ 的 x component 為 $x(t) = r\cos\omega t$] 相差90°

y component 為 $y(t) = r\sin\omega t$

∴ $\vec{r}$ 的 x, y 分量皆為SHM (SHM是UCM在軸上的投影量)

⇒ understand ω in SHM even though there is no angle involved.

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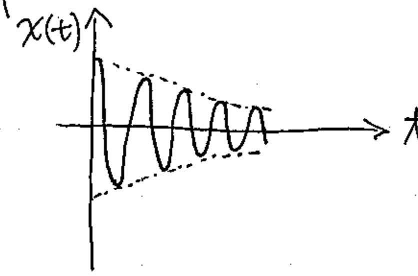
5、Damped and driven 振盪運動

o Damped (阻滯) harmonic motion

In SHM: $\frac{d^2x}{dt^2}+\omega^2 x=0$ is an ideal case in which no energy dissipation.

But in reality, 一定有能量損耗, 如果沒有額外的 energy 補入系統, 則振幅將慢慢減小, 最後終致停止.

⇒ damped oscillation



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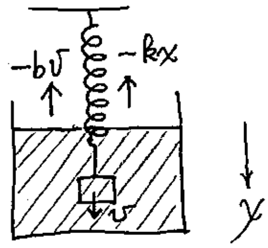

(or spring-system in a fluid.)

一般的 damping force (阻滯力, 用 F_d 表示) $\propto v$ 且與運動方向相反,

∴ $F_d=-bv=-b\frac{dx}{dt}$

where $b>0$ and is ~ damping strength

∴ Newton II: $F_{net}=ma=F_s+F_d=-kx-b\frac{dx}{dt}$

i.e. $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$

其解為 $x(t)=Ae^{-bt/2m}\cos(\omega' t+\phi)$, where

$$\omega'=\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2}=\sqrt{\omega_0^2-\left(\frac{b}{2m}\right)^2}$$

$\omega_0=\sqrt{\frac{k}{m}}$ = natural angular frequency.

ω' 決定 $x(t)$ 的行為

(i) $\omega' \in$ 實數 i.e. $\omega_0>\frac{b}{2m}$, 阻滯強度小時 ⇒ 振幅衰減 + 振盪

如上左圖, 此時為 underdamped.

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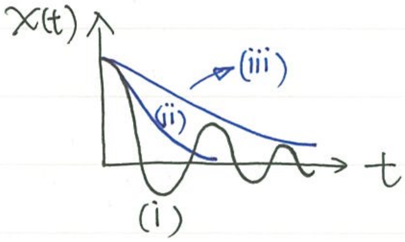
(ii) 阻滯強度增到 $\omega_0=\frac{b}{2m}$，則 $x(t)=A\cos\phi\cdot e^{-bt/2m}$

⇒沒有振盪，且快速到 $x=0$：critical damping、

(iii) 阻滯強度更大，阻滯力主宰運動，沒有振盪且緩慢衰減到 $x=0$



$\omega' \in$ pure imaginary、

(純虛數)

(i) underdamping

(ii) critical damping

(iii) overdamping

○ Driven oscillation (強迫振盪) and resonance (共振)

盪秋遷：

$\vec{F}_{ext}$ on oscillation：driven oscillation (外力作功，輸入能量)

For mass-spring system：$F_{ext}=F_0\cos\omega_d t$, where

ω_d = driving frequency

$$\text{Newton II}: \quad F_{total} = F_s + F_d + F_{ext}$$
$$= -kx - b\frac{dx}{dt} + F_0\cos\omega_d t$$
$$= ma$$
$$= m\frac{d^2x}{dt^2}$$

$$\therefore \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos\omega_d t$$

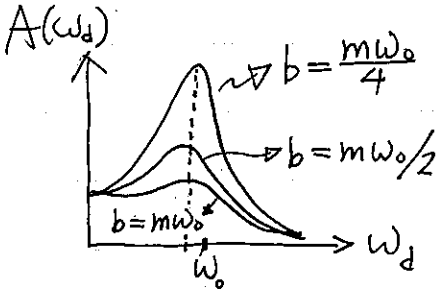
剛開始的運動較複雜，但最後会達到steady-state 振盪，∴

we expect $x(t)=A\cos(\omega_d t+\delta)$ for steady-state。代入微分運動

可、 $\Rightarrow A(\omega_d)=\frac{F_0}{m}\left[(\omega_d^2-\omega_0^2)^2+\frac{b^2}{m^2}\omega_d^2\right]^{-1/2}$

A has max when $\omega_d^2=\omega_0^2-\frac{b^2}{2m^2}$

i.e. $A(\omega_d)$ has max. when ω_d is near ω_0：resonance、 羅



the weaker the damping, the more sharply peaked is the resonance curve.

∴ 在弱阻滯系統可用小驅動力達到大振幅—共振.

實際建物應避免產生共振，如建築物的ω_0應避開地震波的頻率。

Resonance is important in microscopic systems:

magnetron → microwaves heat food

ionize gases.

CO_2 acting like mass-spring system resonates at some of the frequencies of I.R. → greenhouse effect.

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