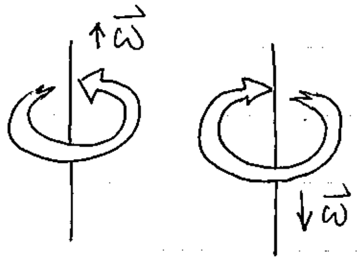
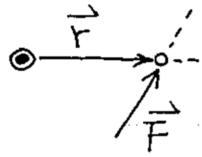
1. $\vec{\omega}$, $\vec{\alpha}$ and $\vec{\tau}$

$\omega = \frac{d\theta}{dt}$, ω的方向? $\Rightarrow$ 右手定則：姆指=方向, 彎曲的四指為轉動方向.

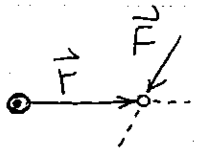


$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t}$, ∴ $\vec{\alpha}$ 是 $\Delta\vec{\omega}$ 的方向, 不是 $\vec{\omega}$ 的方向、

$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$ ：向量外積的方向由右手定則決定.



$\vec{\tau} = \vec{r} \times \vec{F}$, 方向：out of page — ⊙

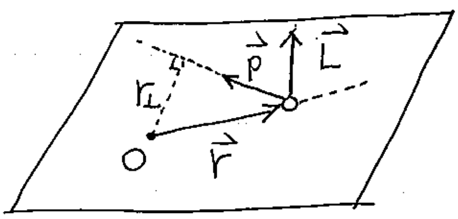


$\vec{\tau} = \vec{r} \times \vec{F}$, 方向：into the page — ⊗

2. Angular momentum (角動量, 以 $\vec{l}$ or $\vec{L}$ 表示)

定義：mass = m, velocity $\vec{v} = \frac{d\vec{r}}{dt}$ 的質點 ($\vec{r}$ = 位置向量)
相對於原點O (或參考點)
的角動量為 $\vec{L}$

$\vec{L} = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p}$



如右圖 $r_\perp = r\sin(\pi - \theta)$
$= r\sin\theta$ = moment arm (~level arm in τ)
θ 為 $\vec{r}$ 和 $\vec{p}$ 的夾角.

羅

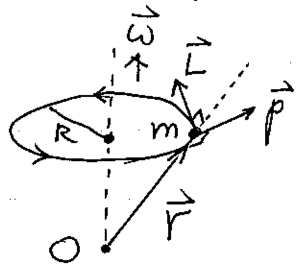
Wolfson ch 11

Similar to $\vec{p} = m\vec{v}$, is $\vec{L} = I\vec{\omega}$?

Yes, in some symmetric objects, like a point mass, wheel or sphere rotating about a fixed axis.

但其他情形不一定成立，$\vec{L}$ 和 $\vec{\omega}$ 甚至不同方向.

例如：



參考點 O 不在 m 運轉的圓心上，∴ $\vec{L}$ 和 $\vec{\omega}$ 不同方向.

$(L = m\omega r R,\ I\omega = m\omega R^2)$

3. Newton II in rotation and 角動量守恆

linear: $\vec{p} = m\vec{v}$ and $\vec{F} = \frac{d}{dt}\vec{p}$ (for point mass) or

$\vec{F}_{ext} = \frac{d}{dt}\vec{P}$ (for system of particles)

Is $\vec{\tau} = \frac{d}{dt}\vec{L}$?

(i) for point mass

$$\vec{L} = \vec{r}\times\vec{p},\quad \therefore \frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r}\times\vec{p})$$
$$= \left(\frac{d\vec{r}}{dt}\right)\times\vec{p} + \vec{r}\times\left(\frac{d\vec{p}}{dt}\right)$$
$$= \vec{v}\times(m\vec{v}) + \vec{r}\times\vec{F}$$
$$= \vec{r}\times\vec{F}$$
$$= \vec{\tau}$$

$\therefore \vec{\tau} = \frac{d\vec{L}}{dt}$ if $\vec{\tau}$ and $\vec{L}$ 具有相同的參考點.

(ii) for system of particles

$$\vec{L} = \Sigma L_i = \Sigma(\vec{r}_i\times\vec{p}_i)$$

$$\therefore \frac{d\vec{L}}{dt} = \Sigma\left(\frac{d\vec{r}_i}{dt}\times\vec{p}_i + \vec{r}_i\times\frac{d\vec{p}_i}{dt}\right) = \Sigma(\vec{r}_i\times\vec{F}_i) = \Sigma\tau_i$$

Similar to the forces in system of particles, the internal torques cancel in pair.

羅

$\because \dfrac{d\vec{L}}{dt} = \vec{\tau}_{ext}$ if $\vec{L}$ and $\vec{\tau}_{ext}$ 具相同的參考點.

○ 角動量守恒

When $\vec{\tau}_{ext} = 0$, 則系統的 $\vec{L} = \text{constant}$, 即系統的角動量守恒, i.e. $L_f = L_i$, or

$I_f \omega_f = I_i \omega_i \Rightarrow$ 溜冰者可依体態改變 I, 達到控制 ω.

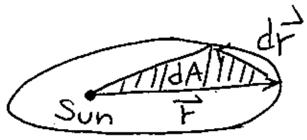
$\Rightarrow$ Kepler's II 行星定律 (面積速率相等定律)

(i) 行星的角動量 $L = \text{constant}$.

以恒星 (e.g. Sun) 為參考點, 行星的 $\vec{r}$ 和 $\vec{F}$ 在同一連線上, $\because \vec{\tau} = \vec{r} \times \vec{F} = 0 = \dfrac{d\vec{L}}{dt}$, $\therefore L = \text{constant}$.

(ii) $\vec{L} = \vec{r} \times \vec{p}$

$= m\vec{r} \times \vec{v} = \text{constant}$



如右圖 $d\vec{r} = \vec{v} \cdot dt$

$$\because dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times (\vec{v} \cdot dt)\right|$$

$$= \frac{dt}{2m}\left|\vec{r} \times (m\vec{v})\right| = \frac{dt}{2m}\left|\vec{r} \times \vec{p}\right| = \frac{dt}{2m}\left|\vec{L}\right|$$

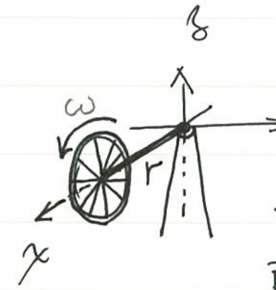
$$\therefore \frac{dA}{dt} = \frac{|\vec{L}|}{2m} = \text{constant.}$$ (也可以用 $\dfrac{\Delta A}{\Delta t}$ 想, $d \to \Delta$)

羅

4. Gyroscopes and Precession

討論之便而已，也可不在x-y平面上。

轉動軸的方向改變，如右圖的gyroscope：飛輪轉動時，轉軸繞z-軸在x-y平面上轉動，即所謂的precession (進動)，此時飛輪不會掉到地上.



$\vec r = r\hat i$

$\vec F = Mg(-\hat k)$

$\therefore \vec\tau = Mgr\hat j$

(i) When 飛輪不轉動時，作用的 $\vec\tau = Mgr\hat j$.
∴飛輪在釋放後即向下掉. ($-\hat k$ 方向)

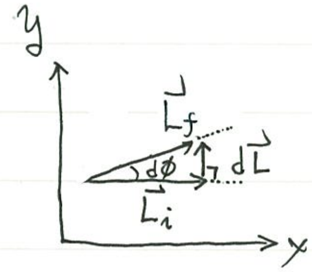
(ii) When 飛輪轉動時，其具有的角動量 $\vec L = I\omega\hat i$，作用的 $\vec\tau$ 也(i)相同，∴ $\vec\tau \perp \vec L$.

但 $\frac{d\vec L}{dt} = \vec\tau$ i.e. $d\vec L = \vec\tau\cdot dt$ 或 $\Delta\vec L = \vec\tau\cdot\Delta t$, i.e. $d\vec L \parallel \vec\tau$

$\Delta\vec L = \vec L_f - \vec L_i = \vec\tau\cdot\Delta t$

也就是 $\vec L$ 的變化量 $d\vec L$ (或 $\Delta\vec L$) 垂直於 $\vec L$

⇒



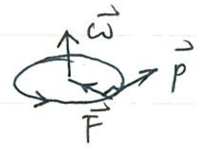
轉軸繞z軸，在x-y平面轉動 ∴ precession

$\vec L_i \to \vec L_f$ in dt

令 precessional angular velocity Ω_p

則 $\Omega_p = \frac{d\phi}{dt} = \frac{|d\vec L|}{|\vec L|}\cdot\frac{1}{dt} = \left|\frac{d\vec L}{dt}\right|\cdot\frac{1}{|\vec L|} = \frac{\tau}{I\omega} = \frac{Mgr}{I\omega} = \frac{\tau}{L}$ ($\Omega_p \ll \omega$)

or $\vec\tau = \vec\Omega_p \times \vec L$ (~ UCM $\vec F_r = \vec\omega \times \vec p$



)

(此地以 $\vec L = I\omega\hat i$ 作說明比較簡單，$\vec L$ 亦可在其他方向，但仍是precession about z-axis)

→ 地軸的precession週期：26,000 years.

雜

1. 靜態平衡(static equilibrium)

平衡=? → the net external force and torque on the body are both zero.

成立的條件：Newton I 運用於平移和轉動

平移：$F_{net}=0=ma=m\frac{dv}{dt} \Rightarrow v=\text{constant}$

轉動：$\tau_{net}=0=I\alpha=I\frac{d\omega}{dt} \Rightarrow \omega=\text{constant}$

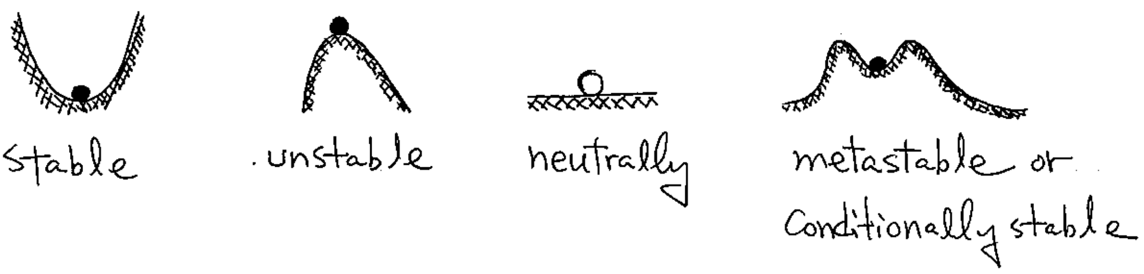
} 在 $v=0$ and $\omega=0$ 為"靜態"平衡。

∴物体處於靜態平衡的條件為

$$\sum \vec{F}_i = 0 = \vec{F}_{net} \quad \text{and} \quad \sum \vec{\tau}_i = 0 = \vec{\tau}_{net}$$

⇒∵處於靜態，∴測量τ可用任何點為參考點(or 支點)

2. 平衡的種類（U-curve 的對應）



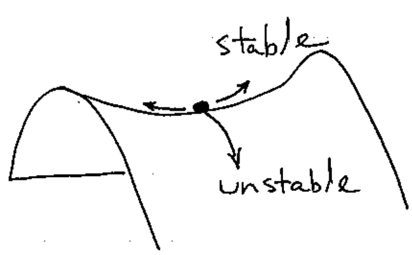
平衡 = $F_{net}=0=-\frac{dU}{dx}$

上列4種平衡狀態皆有 $\frac{dU}{dx}=0$，地形的函數即為U.

stable：$\frac{d^2U}{dx^2}>0$

unstable：$\frac{d^2U}{dx^2}<0$

neutrally：$\frac{d^2U}{dx^2}=0$



3D 的平衡狀態：某一個方向是stable，但另一方向卻為 unstable

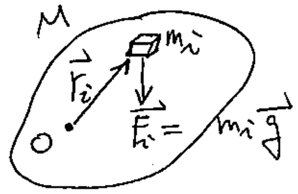
→ saddle-shaped U curve，如上圖.

羅

3. 重心 (Center of gravity)

system of particles (m_i) or

連續体 (dm) of total mass M



$$\vec{\tau} = \sum \vec{\tau_i} = \sum \vec{r_i} \times \vec{F_i}$$

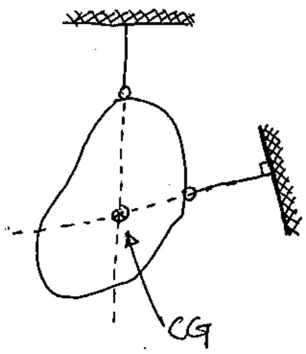
$$= \sum \vec{r_i} \times (m_i\vec{g}) = \left(\sum m_i\vec{r_i}\right) \times \vec{g}$$

$$= \left(\frac{\sum m_i\vec{r_i}}{M}\right) \times M\vec{g}$$

$$= \vec{r}_{CM} \times M\vec{g}$$

$$= \vec{r}_{CG} \times M\vec{g} \qquad \Rightarrow \vec{r}_{CG} = \vec{r}_{CM} \text{ when } \vec{g} \text{ is uniform.}$$

How to find $\vec{r}_{CG}$?



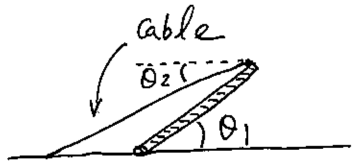
二次懸吊的垂直線

交叉點即為重心位置.

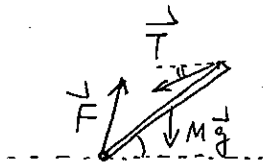
羅

Wolfson 12 Example 12.1

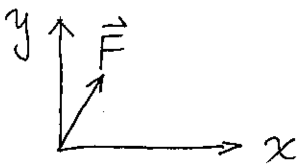
Mass $=M$ 均勻分布，長 L 的 drawbridge，
如右圖拉起時，吊起橋的 cable 張力 $=$?



Free-body diagram of drawbridge:



$\Rightarrow$ 在 $\vec{F}$ 的起點設為 x-y 平面的原點(或支點、參考點)



靜力平衡 $\Leftrightarrow \Sigma\vec{F}=0,\ \Sigma\vec{\tau}=0$

$F_x: F\cos\theta - T\cos\theta_2 = 0$ —(1)
$F_y: F\sin\theta - Mg - T\sin\theta_2 = 0$ —(2)
$\Big\}$ θ 為 $\vec{F}$ 和 x-軸夾角.

$\Sigma\vec{\tau}=0$，$\vec{F}$ 產生的 $\vec{\tau}=0$

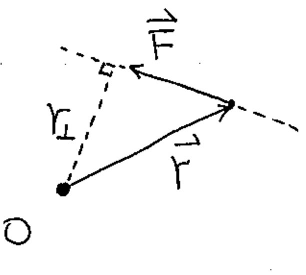
$\vec{T}$ 產生的 $\vec{\tau}$ 方向為 $\hat{k}$, moment arm $r_\perp = L\sin(\theta_1-\theta_2)$

$M\vec{g}$ 產生的 $\vec{\tau}$ 方向為 $-\hat{k}$, moment arm $r_\perp = \frac{L}{2}\cos\theta_1$

$\therefore\ \tau = 0 = T\cdot L\sin(\theta_1-\theta_2) - Mg\cdot\frac{L}{2}\cos\theta_1$ —(3)

From (3) $T = \frac{1}{2}Mg\frac{\cos\theta_1}{\sin(\theta_1-\theta_2)} = \frac{1}{2}Mg\frac{\cos 30^\circ}{\sin 15^\circ} = 180\ kN$

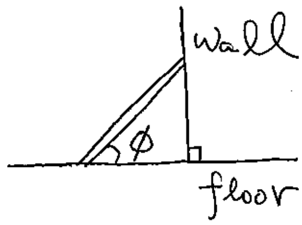
$\vec{F}$ 可求 (1) 和 (2) 求出。



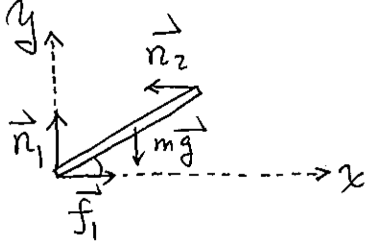
$r_\perp$ = moment arm

羅

Mass m 均勻分布,長L的梯子斜倚在wall設置,如右圖. 若wall無friction,地板則有 μ_1 的靜摩擦係數,則梯子不滑動的最小角度 $\phi=?$



Free-body diagram of ladder



O點在? → 最多力作用的點: 梯-地板接點,
靜摩擦力的方向?
→ 想想若地板沒有friction, ladder滑動的反方向即為靜摩擦力 $\vec{f_1}$ 的方向: +x

$$\therefore \Sigma F_x: f_1 - n_2 = 0 \quad -(1)$$
$$\Sigma F_y: n_1 - mg = 0 \quad -(2)$$

$\vec{n}_2$ 產生的 τ 方向為 $\hat{k}$, moment arm $= L\sin\phi$

$m\vec{g}$ 〃 $-\hat{k}$, $= \frac{L}{2}\cos\phi$

$$\therefore \Sigma\tau = 0 = n_2 L\sin\phi - mg\cdot\frac{L}{2}\cos\phi \quad —(3)$$

From (3): $\tan\phi = \frac{mg}{2n_2}$, $n_2 = f_1 = \mu_1 n_1 = \mu_1 mg$ [由(1)、(2)].

$$\therefore \tan\phi = \frac{1}{2\mu_1}.$$

⇒ Problem 35: 地板 μ_1, wall μ_2, 則 min. $\phi = ?$

$\vec{f_2}$: +y 方向, 產生的 τ 為 $\hat{k}$, moment arm $= L\cos\phi$

$$\therefore \Sigma F_x: f_1 - n_2 = 0 \quad —(4)$$
$$\Sigma F_y: n_1 - mg + f_2 = 0 \quad —(5)$$

} Note: $f = \mu n$.

$$\Sigma\tau: n_2 L\sin\phi - mg\cdot\frac{L}{2}\cos\phi + f_2\cdot L\cos\phi = 0 \quad —(6)$$

From (6) $\tan\phi = \frac{mg}{2}\frac{1}{n_2} - \mu_2$, 又 from (4)、(5) $n_2 = \frac{\mu_1}{1+\mu_1\mu_2}mg$

$$\therefore \tan\phi = \frac{1-\mu_1\mu_2}{2\mu_1}$$

羅

e^- 的 $U(x) = ax^2 - bx^4$ where a, b are positive constants, Find e^- 的平衡位置 and 其 stability.

平衡位置：$F=0$ 的位置. 又 $F = -\frac{dU}{dx}$

$\therefore$ 平衡位置 = U curve 上斜率為零 (水平線) 的位置.

$$\frac{dU}{dx} = 0 = 2x(a - 2bx^2)$$

$\therefore x = 0$ 及 $x = \pm\sqrt{\frac{a}{2b}}$ 為平衡位置.

平衡位置的 stability 由 $\frac{d^2U}{dx^2}$ 決定：

$$\frac{d^2U}{dx^2} = \frac{d}{dx}\left(\frac{dU}{dx}\right) = 2a - 12bx^2$$

$\therefore$ when $x = 0$, $\frac{d^2U}{dx^2} = 2a > 0 \Rightarrow$ stable

when $x = \pm\sqrt{\frac{a}{2b}}$, $\frac{d^2U}{dx^2} = -4a < 0 \Rightarrow$ unstable