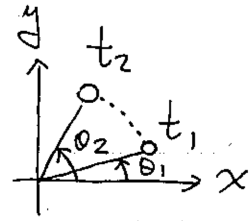
Wolfson Ch 10 Rotational Motion

1、角速度與角加速度

◦ ~平移運動，定義轉動



Angular displacement：$\Delta\theta = \theta_2 - \theta_1$

$(\sim \Delta x = x_2 - x_1)$

$\Delta\theta$的方向：逆時針(counterclockwise)轉動：>0 → (ccw)

順時針(clockwise)轉動：<0 → cw

右手定則：$x \to y$ 的方向為 $+z$，$y \to x$（順時針）為 $-z$

∴ 平均角速度(average angular velocity) $\bar{\omega} = \dfrac{\theta_2-\theta_1}{t_2-t_1} = \dfrac{\Delta\theta}{\Delta t}$，$(\sim \bar{v} = \dfrac{\Delta x}{\Delta t})$

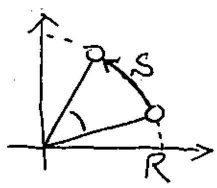
角速度 $\omega = \lim\limits_{\Delta t\to 0} \bar{\omega} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$ $(\sim v = dx/dt)$，$[\omega] = s^{-1}$

角加速度 $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ $\left(a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}\right)$，$[\alpha] = s^{-2}$

◦

linear vs. angular variables

弧長 $s = R\theta$（θ 用 radian）



linear speed $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(R\theta)$

$= R\dfrac{d\theta}{dt} = R\omega$

週期(period) T = 轉一圈的時間，∴ $\omega = \dfrac{2\pi}{T}$ or

$T = \dfrac{2\pi R}{v} = \dfrac{2\pi R}{R\omega} = \dfrac{2\pi}{\omega}$.

f：頻率(frequency) = 每 sec 轉動的圈數，∴ $f = \dfrac{1}{T}$ or

$\omega = 2\pi f = \dfrac{2\pi}{T}$，$[f] = s^{-1}$。 f：rpm = 每分鐘轉的圈數.

revolutions per minute

Note：f 與 ω 的單位雖相同：s^{-1}，但代表的意義不同、

$[\omega]$ 應是 rad·s^{-1}，但 rad 沒單位。

v 是切線(tangent)速率 $\Rightarrow \frac{dv}{dt}$ = 切線加速度 a_t (用於 change speed)

$\therefore a_t = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$

而徑向(radial)方向的加速度
即向心加速度(用於 change 方向) $a_r = \frac{v^2}{R} = R\omega^2$

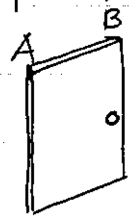
total 的加速度 $\vec{a} = \vec{a_t} + \vec{a_r} = a_t\hat{\theta} + a_r\hat{r}$

Table 10.1 <u>Linear vs. Angular quatities</u>

2. Torque and Newton II for rotation

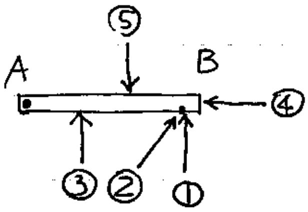
$x \sim \theta$, $v \sim \omega$, $a \sim \alpha$, $F \sim ?$ → 描述轉動的難易程度需有作用力.

轉動門的難易程度



A 為轉軸

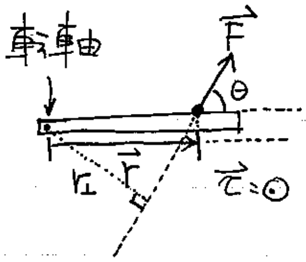
top view



用大小相同的力作用在
(i) 不同位置 (ii) 不同方向 ~~如左~~

轉動難易:① > ② > ③ > ④ (不動) ; ①~③:CCW, ⑤:CW

定義 torque (用 τ 表示) $\vec{\tau} = \vec{r} \times \vec{F}$



$\therefore \tau = |\vec{r}| \times |\vec{F}| \times \sin\theta$

$= F \cdot r \cdot \sin\theta$

$= F \cdot r_\perp$

$r_\perp$: level arm $= r\sin\theta$

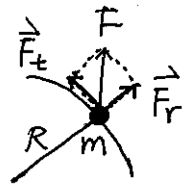
右手定則定 $\vec{\tau}$ 的方向

2D: ⊙ out of page

⊗ into the page

∘ Newton II for linear : $F = ma$

for angular : $\tau = \underline{?} \times \alpha$



For the simplest case : 質點 m

$$F_t = m a_t$$

$$\tau = R F_t = m R a_t = m R \cdot R\alpha = m R^2 \cdot \alpha$$

$$\equiv I\alpha \quad (\text{Newton II for rotation}), \text{ where } I = mR^2$$

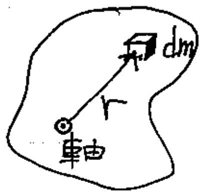
I : 轉動慣量 (rotational inertia or moment of inertia)

∴ system of particles : $I = \sum m_i r_i^2$

r_i : m_i 到轉軸的距離。

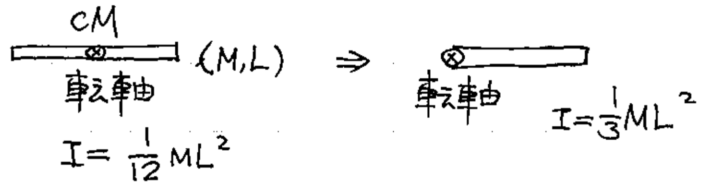
∴ for 連續体 : $I = \int r^2 dm$

(～CM 的作法) ⇒ 例題



∘ 平行軸 (parallel-axis) 定理 for I

例子 :



兩平行轉軸相距 d，其中之一通過CM形式的轉動慣量為 I_{CM}

則另一軸的 $I = I_{CM} + Md^2$，M為連續体之mass.

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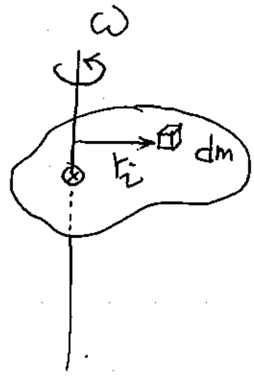
3、Rotational energy

○ K

Consider 一連續体繞一轉軸轉動 ω
連續切割成 dm, 每個 dm 到轉軸的
距離為 r_i, 則 dm 形成的動能為



$$dK = \frac{1}{2}(dm)(v^2) = \frac{1}{2}(dm)(r_i\omega)^2$$
$$= \frac{1}{2}\omega^2 \cdot r_i^2 \cdot dm$$

$\therefore$ total 動能 $K = \int dK = \frac{1}{2}\omega^2 \int r_i^2\, dm = \frac{1}{2} I \omega^2$

rotational $K = \frac{1}{2} I \omega^2$

○ Work in rotational motion

work-energy theorem: $W = \Delta K$

In rotation, $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta \quad (\sim W = \int_{x_1}^{x_2} F(x)\, dx)$

$$\Delta K_{rot} = \frac{1}{2} I (\omega_f^2 - \omega_i^2)$$

If τ is constant, $W = \tau(\theta_f - \theta_i)$

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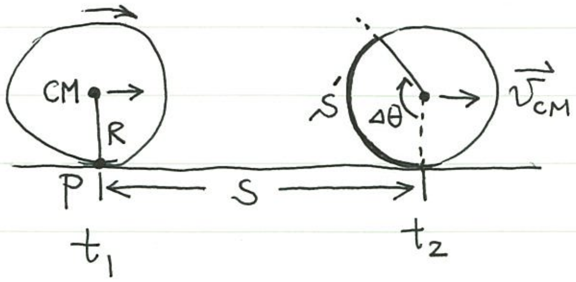
4. Rolling motion

∘ Rolling

Rolling：$S' = S$，有滑動則 $S' > S$ or $S' < S$
slipping



$S = S' = R \cdot \Delta\theta$

$\omega = \Delta\theta / \Delta t$

(i) 滾輪轉動的切線速率 $v = R\omega$

(ii) Δt 內，CM移動的距離 $= S$，∴滾輪的

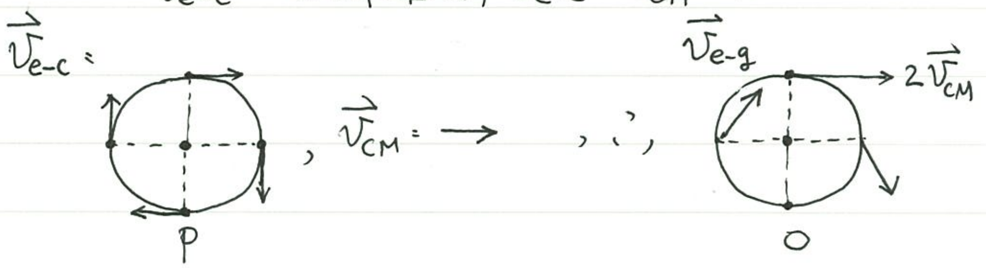
$$v_{CM} = \frac{S}{\Delta t} = R\frac{\Delta\theta}{\Delta t} = R\omega \text{ (向右)}$$

(iii) 滾輪邊緣上的點相對於地的速度 = ?

e = edge points，c = center of mass，g = ground

$$\vec{v}_{e-g} = \vec{v}_{e-c} + \vec{v}_{c-g}\text{，其中 } \vec{v}_{c-g} = \vec{v}_{CM}$$

$\vec{v}_{e-c}$ = 切線速度，$v_{e-c} = v_{CM} = R\omega$



v_{e-g} 在top點最大 $= 2v_{CM}$

在接地點P為0.

∘ K of rolling

地面觀察者所測之 K = 以CM為軸的轉動 + CM的平移運動

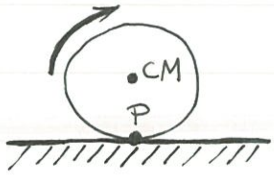
∴ $K = K_{rot} + K_{transl} = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$

since $v_{CM} = R\omega$

∴ $K = \frac{1}{2}(I_{CM} + MR^2)\omega^2 = \frac{1}{2} I_P \omega^2$ (平行軸定理)

where I_P：以接地點P為軸的轉動慣量.

○ Rolling and friction



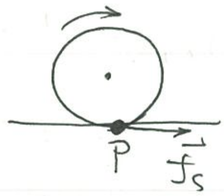
(i) No slipping 時，
接地點P对地是靜止的，∴ 若有 friction，則
在此點的 friction 是 static ⇒ friction 不作功（位移為0）.
⇒ 能量守恒 $\Delta E = 0$ works！ （$\Delta E = W_{nc}$）

no slipping，$v_{CM} = R\omega$，∴ $\frac{d}{dt}v_{CM} = a_{CM} = R\frac{d\omega}{dt} = R\alpha$

∴ $\boxed{a_{CM} = R\alpha}$ for rolling.

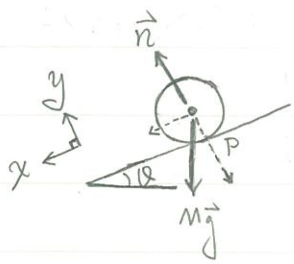
如有 slipping，則 $a_{CM} \neq R\alpha$.

(ii) 靜摩擦力 $\vec{f_s}$（用"防止滑動"想）
When $a_{CM} > 0$ and no slipping，
加速往右 rolling，為防止產生
slipping，$\vec{f_s}$ 作用在滾輪的
接地點處為向右方向.



if $a_{CM} < 0$，then $\overset{\vec{f_s}}{\longleftarrow}$ P

(iii) 斜坡上的滾輪 M，如要產生 pure rolling，則 $\vec{f_s} = ?$



→ Think：M 滑動下斜坡。為使滑動不產生，
作用在M的 $\vec{f_s}$ 方向必需是 ↗.

重力 $M\vec{g}$ 及 $\vec{n}$ 作用在 CM，level arm $r_\perp = 0$
对 M 的 $\tau = 0$。惟一的 τ 來自 f_s.

$$\left.\begin{aligned} &\therefore \tau = Rf_s = I_{cm}\alpha = I_{cm}a_{cm}/R \quad -(1) \\ &\text{Newton II in } x: Mg\sin\theta - f_s = Ma_{cm} \quad -(2) \end{aligned}\right\} f_s = \frac{Mg\sin\theta}{1 + \frac{MR^2}{I_{cm}}}.$$

Wolfson Example 10.2

風力発電机的blade (扇葉) 長28m,
有風時,以21 rpm転動,風停時,以
等角加速度 0.12 rad/s² 減慢転動,則
在停止前,共転多少圈?

$\sim \ v_f^2 = 0 = v_i^2 + 2a \cdot \Delta x$

$\omega_f^2 - \omega_i^2 = 2\alpha \cdot (\Delta\theta)$

$\omega_f = 0$, $\omega_i = 21 \text{ rpm} = 21 \times \frac{2\pi}{60}$ rad/s , and

$\alpha = -0.12 \text{ rad/s}^2$

$\therefore \ 0 - \left(\frac{21 \times 2\pi}{60}\right)^2 = 2 \cdot (-0.12) \cdot \Delta\theta$

$\Rightarrow \Delta\theta = 20 \text{ rad} = \frac{20}{2\pi} = 3.2$ revolutions

or change the unit of α

$\alpha = -0.12 \text{ rad} \cdot s^{-2} = -0.12 \cdot \frac{\text{revolution}}{2\pi} \cdot \left(\frac{1}{60}\text{min}\right)^{-2}$

$= -0.12 \times \frac{1}{2\pi} \times 60^2 \cdot \text{revolution} \cdot \text{min}^{-2}$

then $0 - 21^2 = -2 \times 0.12 \times \frac{1}{2\pi} \times 60^2 \times (\Delta\theta)$ here $[\Delta\theta] = $ revolution

$\therefore \ \Delta\theta = 3.2$ revolutions.

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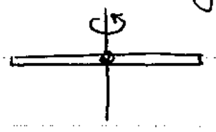
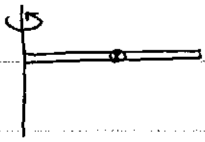
Wolfson Example 10.5 : 1D I

uniform 1D rod, long L and mass M

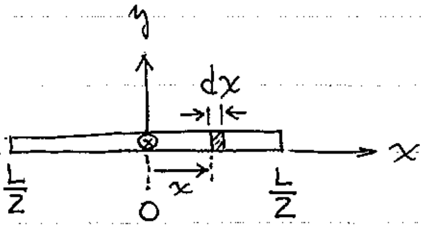
(i) 

$I_{CM} = ?$

(ii) 

$I = ?$

(i)



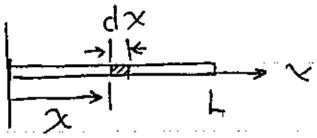
切割成長度為 dx 的 dm

此 $dm = \lambda \cdot dx = \frac{M}{L} \cdot dx$

$$\therefore I_{CM} = \int_{-L/2}^{L/2} x^2 \cdot dm = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{2M}{L} \int_0^{L/2} x^2 \cdot dx = \frac{2M}{L} \times \frac{1}{3} \times \left(\frac{L}{2}\right)^3$$

$$= \frac{1}{12} M L^2$$

(ii) Similar to (i)



$$I = \int_0^L x^2 \cdot dm = \frac{M}{L} \int_0^L x^2 \cdot dx = \frac{1}{3} M L^2 = M \cdot \left(\frac{L}{2}\right)^2 + \frac{1}{12} M L^2$$

$$= M d^2 + I_{CM}$$ —— 平行軸定理.

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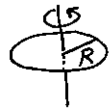
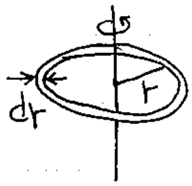
Wolfson Example 10.7 : 2D I

Uniform disk of radius R and mass M, $I_{CM}=?$



將 disk 分割成不同半徑 r, 但相同寬度 dr 的同心 ring = dm



ring的面積 $dA=$ 長 × 寬 $=2\pi r\cdot dr$

則 dm 形成的轉動慣量

$$dI=r^2 dm=r^2\cdot\sigma\cdot dA$$

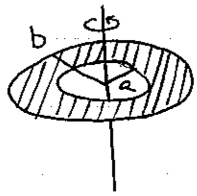
σ: 單位面積的mass $=\frac{M}{\pi R^2}$

$$=r^2\cdot\frac{M}{\pi R^2}\cdot 2\pi r\cdot dr$$

$$=\frac{2M}{R^2}r^3\cdot dr$$

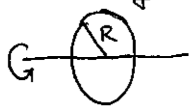
$$\therefore I=\int dI=\frac{2M}{R^2}\int_0^R r^3\cdot dr=\frac{1}{2}MR^2$$

Try →

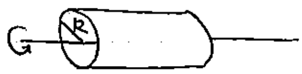


$I_{CM}=\frac{1}{2}M(a^2+b^2)$

1D ring

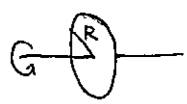


$I=MR^2$



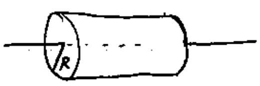
$I=MR^2$

2D disk



$I=\frac{1}{2}MR^2$

cylinder



$I=\frac{1}{2}MR^2$

⇒ 徑轉軸方向延伸長度形成的 3D 物体, 和 2D 具相同的 I

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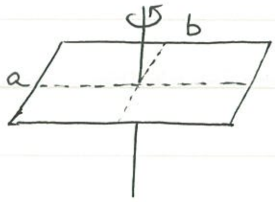
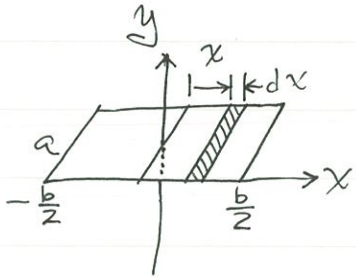
Wolfson Extra Example of 2D I

如 Table 10.2 的 2D 長方形

具 uniform mass M, 則 $I_{CM} = ?$



斜線的 $dm = \sigma \cdot dA$



$\sigma = \frac{M}{ab}$

$$\therefore dm = \frac{M}{ab} \cdot a \cdot dx = \frac{M}{b} \cdot dx$$

$$dI_{CM} = \frac{1}{12} dm \cdot a^2 \quad \text{and} \quad dI = dI_{CM} + dm \cdot x^2 \ (\text{平行軸定理})$$

$$\therefore \ I = \int dI = \int_{-b/2}^{b/2} \left(\frac{1}{12} \cdot a^2 \cdot \frac{M}{b} \cdot dx + \frac{M}{b} \cdot dx \cdot x^2 \right)$$

$$= \frac{M}{12} (a^2 + b^2)$$ (增加厚度,仍具相同的 I)

Note: As $a \to 0$ or $b \to 0 \Rightarrow$ 1D rod.

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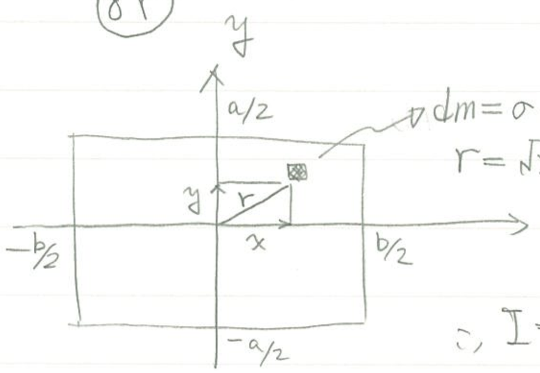
or

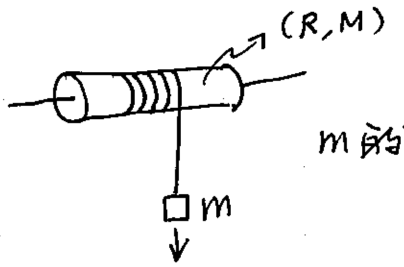


$dm = \sigma \cdot dA = \frac{M}{ab} \cdot dx \cdot dy$

$r = \sqrt{x^2 + y^2}$

$$\therefore I = \int r^2 dm = \frac{M}{ab} \int (x^2 + y^2)\, dx\, dy = \frac{1}{12}(a^2 + b^2)$$

Wolfson Example 10.9



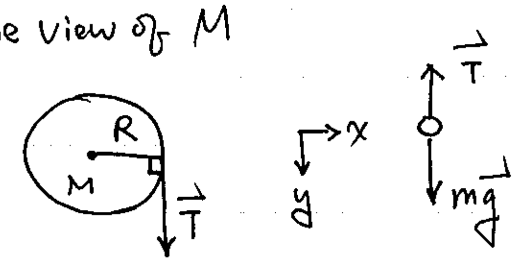
m的下降 $a = ?$ (在沒有滑動的情形下)

Key: m的下降加速度 = M轉動時的切線加速度 a_t

$$\therefore a_t = a = R\alpha = R \cdot \frac{\tau}{I} \quad \text{—①}$$

free-body diagrams of M and m

side view of M



$$\Rightarrow mg - T = ma \quad \text{—②}$$

$$\Downarrow$$

$$\tau = RT$$

$$\therefore ① \Rightarrow a = R \cdot \frac{\tau}{I} = R \cdot \frac{RT}{\frac{1}{2}MR^2} = \frac{2}{M}T \quad \text{代入②}$$

$$mg = T + ma = \left(\frac{M}{2} + m\right)a$$

$$\therefore a = \frac{mg}{\frac{M}{2} + m} = \frac{2mg}{M + 2m} \quad (\text{向下}).$$

Assess: check by $M = 0$, $M = 0$ 時則無轉動量, $\therefore a$ 應該 $= g$.

$\Rightarrow$ Yes.

Note: R 沒有出現在 $a = \frac{2mg}{M+2m}$ 中.

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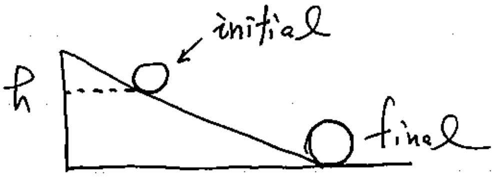
Wolfson Example 10.12 and 觀念例題 1.

實心球(M,R)靜止從h高度的斜坡滾下，則在坡底球的speed=?



E守恆，∴ $E_f = E_i \Rightarrow \Delta K + \Delta U = 0$

$\Delta K = K_f - K_i = K_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$\Delta U = 0 - Mgh = -Mgh$

$\therefore \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$, $I = \frac{2}{5}MR^2$ (for sphere), and $\omega = \frac{v}{R}$

$\therefore \frac{1}{2}Mv^2 + \frac{1}{2}\times\frac{2}{5}MR^2\times\frac{v^2}{R^2} = Mv^2\left(\frac{1}{2}+\frac{1}{5}\right) = \frac{7}{10}Mv^2 = Mgh$

$\therefore v = \sqrt{\frac{10}{7}gh}$ (和M, R無關)

Note：無friction下滑的 $v=\sqrt{2gh}$，會有滾動的物体，其在坡底的speed是小於此值.

觀念例題：滾動比賽，誰先到達坡底?

$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ = CM的平移K + 轉動K

$= \frac{1}{2}Mv^2 + \frac{1}{2}I\cdot\frac{v^2}{R^2}$

∴ 坡底的v和I成反向關係

$I\uparrow, v\downarrow$

例：空心球殼 $I = \frac{2}{3}MR^2 \Rightarrow v = \sqrt{\frac{6}{5}gh}$

實心圓柱 $I = \frac{1}{2}MR^2 \Rightarrow v = \sqrt{\frac{4}{3}gh}$

空心圓柱 $I = MR^2 \Rightarrow v = \sqrt{gh}$

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