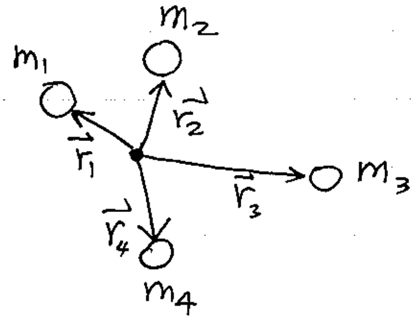
Wolfson Ch 9 Systems of Particles

1. Center of mass (用CM表示)

質點 → 質點系統：rigid body 及非 rigid bodies.
物体的運動可以用CM描述。



CM的定義：
如右图，四個間距固定的質點系統，CM的位置為

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \sum_i m_i\vec{r}_i \Big/ \sum_i m_i$$

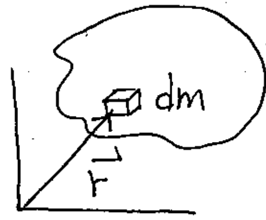
$$= \frac{1}{M}\sum m_i\vec{r}_i$$

$$= x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $x_{CM} = \frac{1}{M}\sum m_i x_i$. $y_{CM} \sim \cdots$, $z_{CM} \sim \cdots$.

For solid body：
切割成 dm and let $dm \to 0$.
類似上面的定義



$$x_{CM} = \frac{\int x\,dm}{\int dm} = \frac{1}{M}\int x\,dm$$

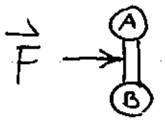
$$\Rightarrow \quad \vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm \qquad \text{and } dm \to \text{density} \times \text{volume}$$

$\boxed{\text{Density}}$ 1D：$\lambda = \frac{dm}{d\ell}$ = 單位長度的質量 = linear density, ∴ $[\lambda] = kg \cdot m^{-1}$.
2D：$\sigma = dm/dA$ = 單位面積的質量 = area density, ∴ $[\sigma] = kg \cdot m^{-2}$.
3D：$\rho = dm/dV$ = 單位体積的質量 = density, ∴ $[\rho] = kg \cdot m^{-3}$

∴ $dm = \lambda \cdot d\ell$ or $\sigma \cdot dA$ or $\rho \cdot dV$

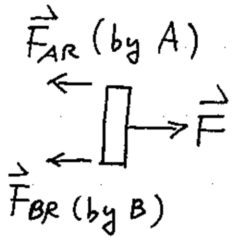
Symmetry for $\vec{r}_{CM}$!

External force vs. internal forces in a system of particles

例: 

Ⓐ → $\vec{F}_{RA}$ by rod,

Ⓑ → $\vec{F}_{RB}$ by rod,



But for the whole system (Ⓐ + Rod + Ⓑ), $\vec{F}_{RA}$, $\vec{F}_{RB}$, $\vec{F}_{AR}$ and $\vec{F}_{BR}$ are internal forces, the only external force is $\vec{F}$.

$$\vec{F}_{total} = \sum \vec{F}_{external} + \sum \vec{F}_{internal}, \text{ but } \sum \vec{F}_{internal} = 0 \text{ by Newton III.}$$

$$\Rightarrow \vec{F}_{net,ext} = \sum \vec{F}_{external} = M\vec{a}_{CM} = M\frac{d^2}{dt^2}\vec{r}_{CM}$$

2. 質點系統的動量 (用P表示)

For 單一質點 $\vec{p} = m\vec{v}$

For 質點系統 $\vec{P} = \sum \vec{p}_i = \sum m_i \vec{v}_i = \sum m_i \frac{d\vec{r}_i}{dt}$

$$= \frac{d}{dt}\sum m_i \vec{r}_i = \frac{d}{dt}(M\vec{r}_{CM}) = M\frac{d}{dt}\vec{r}_{CM} = M\vec{v}_{CM}$$

where $\vec{v}_{CM} = \frac{d}{dt}\vec{r}_{CM}$ and M is fixed.

$$\Rightarrow \frac{d}{dt}\vec{P} = \frac{d}{dt}(M\vec{v}_{CM}) = M\vec{a}_{CM} = \vec{F}_{net,ext} \text{ once } M \text{ is fixed.}$$

推論: When $\vec{F}_{net,ext} = 0$ 則 $\frac{d\vec{P}}{dt} = 0$, i.e. $\vec{P}$ = constant. —— 質點系統的線(linear)動量守恆, 不論質點的運動形式為何.

羅

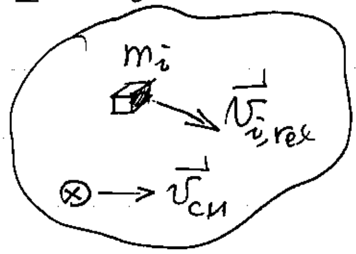
○ 质點系統的 K

$\vec{r}, \vec{v}, \vec{a}, \vec{p}$ 皆向量, 在质點或质點系統有相同的形式.

純量 K or U 则如何?

K:

total K of 质點系統 $K = \Sigma K_i = \Sigma \frac{1}{2} m_i v_i^2$



Let $\vec{v}_i = \vec{v}_{CM} + \vec{v}_{i,rel}$ —— (a)

$\vec{v}_{i,rel}$ = m_i 相对於 CM 的速度.

$$\therefore K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\vec{v}_{CM} + \vec{v}_{i,rel}) \cdot (\vec{v}_{CM} + \vec{v}_{i,rel})$$
$$= \frac{1}{2} m_i v_{CM}^2 + \Sigma m_i \vec{v}_{CM} \cdot \vec{v}_{i,rel} + \frac{1}{2} m_i v_{i,rel}^2$$

$$\therefore K = \Sigma K_i = \frac{1}{2} v_{CM}^2 \Sigma m_i + \vec{v}_{CM} \cdot \Big(\frac{d}{dt} \underbrace{\Sigma m_i \vec{r}_{i,rel}}_{=0}\Big) + \Sigma \frac{1}{2} m_i v_{i,rel}^2$$
$$= \frac{1}{2} M v_{CM}^2 + \Sigma \frac{1}{2} m_i v_{i,rel}^2$$
$$= K_{CM} + K_{rel}$$

$$\Rightarrow \left(\begin{aligned} \vec{r}_i &= \vec{r}_{CM} + \vec{r}_{i,rel} \quad (\times \Sigma m_i) \\ \Sigma m_i \vec{r}_i &= \vec{r}_{CM} \Sigma m_i + \Sigma m_i \vec{r}_{i,rel} \\ M \vec{r}_{CM} &= \vec{r}_{CM} \cdot M + \Sigma m_i \vec{r}_{i,rel} \\ \therefore \Sigma m_i \vec{r}_{i,rel} &= 0 \end{aligned} \right)$$

Where K_{rel} is called internal K.

3. 碰撞與衝量 (Impulse, 用 J 表示)

碰撞物体可大可小, 時間有长有短, 有 2 個特徵

1°: 與碰撞物体的運動狀態相比, 碰撞的交互作用時間極短.

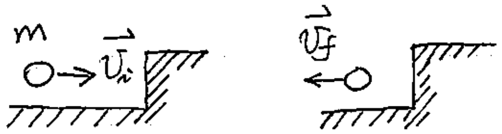
2°: 碰撞的交互作用力強烈。但為系統的 internal forces.

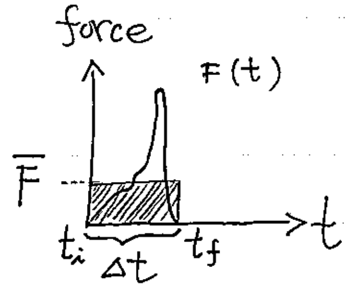
$\Rightarrow$ 碰撞系統的總動量守恆。

羅

○ 衝量 ($\vec{J}$)

碰撞作用力強而短促且形式不明 ⇒ F(t)





$$\vec{J} \equiv \Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,dt \quad \left(\because \frac{d\vec{p}}{dt} = \vec{F}\right)$$

$= \vec{F}(t)$ 在 $[t_i, t_f]$ 與 t-軸間的面積

∴ 碰撞過程可用一個平均力 $\bar{F}$ 得到相同的 J

$$\Rightarrow J = \int_{t_i}^{t_f} F(t)\,dt = \bar{F}\cdot(t_f - t_i) = \bar{F}\cdot\Delta t$$

○ Collision：動量守恆

- elastic：$\Delta K = 0$ (K守恆)
- inelastic：K不守恆
  - (special case) totally inelastic：1+1=1

• 1D 彈性碰撞：

Before： m_1 ○→ v_1 ， m_2 ○→ v_2

After： m_1 ○→ v_1' ， m_2 ○→ v_2'

Σp_x： $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$

∴ $m_1(v_1 - v_1') = m_2(v_2' - v_2)$

ΣK： $m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$

$v_2 - v_1 = -(v_2' - v_1')$ for all 1D 彈性。

(與 m_1, m_2 無關)

Case(1)： $m_1 = m_2$

$$\begin{cases} v_1 + v_2 = v_1' + v_2' \\ v_2 - v_1 = -(v_2' - v_1') \end{cases} \Rightarrow \begin{cases} v_1' = v_2 \\ v_2' = v_1 \end{cases}$$

碰撞後，速度交換。

羅

Case (ii): $m_1 \neq m_2$ and $v_2 = 0$ (stationary target)

$$\left.\begin{aligned} \Sigma p:\ & m_1 v_1 = m_1 v_1' + m_2 v_2' \\ & -v_1 = -(v_2' - v_1') \end{aligned}\right\} \quad v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \ ; \ v_2' = \frac{2m_1}{m_1 + m_2} v_1$$

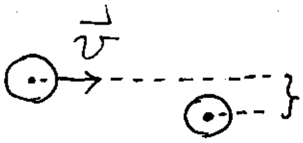
狀況A: $m_1 \gg m_2$ (鉛球撞靜止的乒乓球)

$\Rightarrow v_1' \approx v_1,\ v_2' \approx 2v_1$

狀況B: $m_1 \ll m_2$ (乒乓球撞靜止鉛球)

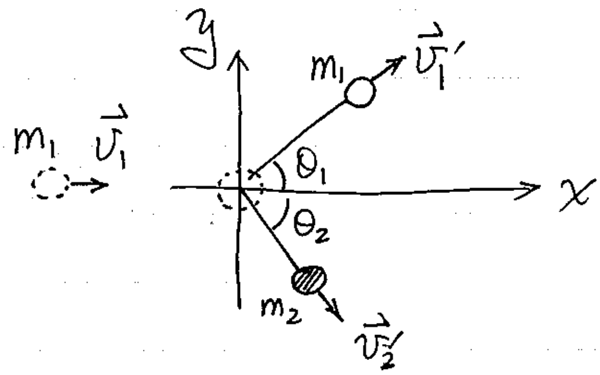
$\Rightarrow v_1' \approx -v_1,\ v_2' \approx 0$

• 2D 彈性碰撞



} b: impact parameter

⇒ 決定碰撞後的方向分佈



(why $v_2 = 0$? ⇒ 自由選擇 frame)

⇒ 各維度的動量守恆 ($\vec{p}$ 守恆) and K 守恆

$$\left.\begin{aligned} \Sigma p_x:\ & m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \\ \Sigma p_y:\ & 0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2 \\ \Sigma K:\ & m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \end{aligned}\right\}$$

4 未知數: $v_1', v_2', \theta_1, \theta_2$

∴ 測出其中的一個,就可知道其他 3 個.

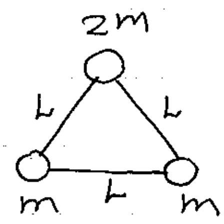
Special case: prove that if $m_1 = m_2$, then $\theta_1 + \theta_2 = \pi/2$.

See Example 9.11

羅

Wolfson Example 9.2

右圖的質點系統，忽略 rod 的 mass
求其 $\vec{r}_{CM}=?$



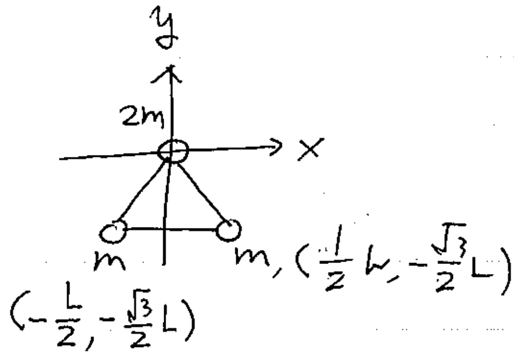
→ 2D frame

→ free to choose coordination system and origin.

How?

⇒ symmetry 軸為其中的一軸.

choose origin at 2m



对稱軸 = y 軸，∴ $x_{CM}=0$

$$y_{CM}=\frac{1}{2m+m+m}(my_1+my_2)$$

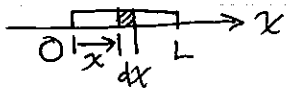
$$=\frac{1}{4}\left(-\frac{\sqrt{3}}{2}L-\frac{\sqrt{3}}{2}L\right)=-\frac{\sqrt{3}}{4}L$$

∴ $\vec{r}_{CM}=L\left(0,-\frac{\sqrt{3}}{4}\right)$ ⇒ 在 2m 正下方 $\frac{\sqrt{3}}{4}L$ 處

羅

Wolfson Extra examples of CM : 1D

1. 長L, mass = M 的 1D rod，



(a) if mass 均勻分布, 即 $\lambda = \frac{M}{L} = \text{constant}$, 則 $x_{CM} = ?$

(b) if mass 非均勻分布如 $\lambda(x) = ax^2$, 則 $a = ?$ and $x_{CM} = ?$

(a) $x_{CM} = \frac{L}{2}$ of course. But we try it by using dm

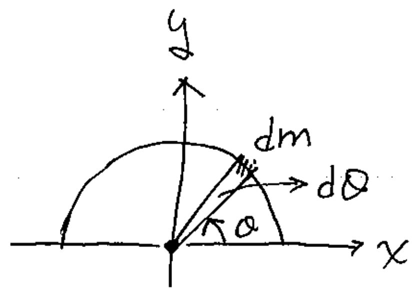
$$\therefore x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\cdot\lambda\cdot dx = \frac{1}{M}\cdot\lambda\int_0^L x\,dx = \frac{1}{M}\cdot\frac{M}{L}\cdot\frac{L^2}{2} = \frac{L}{2}.$$

(b) $$\int dm = M = \int_0^L \lambda(x)\,dx = a\int_0^L x^2\cdot dx = a\cdot\frac{L^3}{3} \quad \therefore a = \frac{3M}{L^3}$$

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\cdot\lambda(x)\,dm = \frac{1}{M}\cdot\int_0^L x\cdot\frac{3M}{L^3}x^2\,dx$$

$$= \frac{3}{L^3}\cdot\frac{1}{4}L^4 = \frac{3}{4}L$$

2. 半徑R, mass = M 的 1D 半圓, 其 $\vec{r}_{CM} = ?$
(mass 均勻分布) (1D rod 形成的半圓)



if 1D半圓如右圖所示
⇒ 對稱軸為通過圓心的垂直軸

$\therefore x_{CM} = 0$

切割成 dm, 則 $dm = \lambda\cdot$弧長 $= \lambda\cdot ds = \lambda\cdot R\,d\theta$
where $\lambda = \frac{M}{L} = \frac{M}{\pi R}$

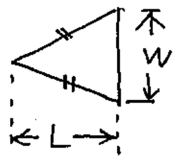
$$\therefore y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int y\cdot\frac{M}{\pi R}\cdot R\,d\theta = \frac{1}{M}\cdot\frac{M}{\pi R}\cdot R\int_0^\pi R\sin\theta\,d\theta$$
$$= \frac{R}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{2R}{\pi}.$$

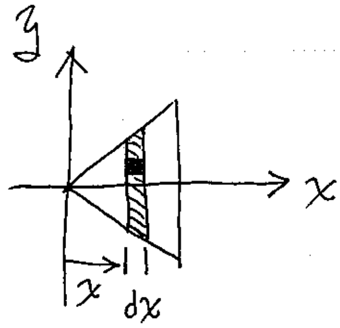
$\therefore$ CM 在圓心上方 $\frac{2R}{\pi}$ 處。

羅

Wolfson Example 9.3

2D連續体的CM：
如右圖可忽略厚度的等腰薄片△，
mass = M，均匀分布，則 $\vec{r}_{CM}$ = ?





⇒ 对稱軸为水平軸 x
∴如右圖所选之 frame 時，$y_{CM} = 0$

切割成垂直條狀的 $dm = \sigma \cdot dA$
σ = area density $= \frac{M}{面積} = \frac{2M}{LW}$

dA = 斜線区域面積 = 寬 × 长 = $dx \cdot h$，此處 $h = \frac{W}{L} x$
h = 在 x 處长條 dA 的长度

$$\therefore dm = \sigma \cdot dA = \frac{2M}{LW} \cdot \frac{W}{L} x \cdot dx = \frac{2M}{L^2} x \cdot dx$$

$$x_{CM} = \frac{1}{M} \int_0^L x \, dm = \frac{1}{M} \cdot \frac{2M}{L^2} \int_0^L x^2 dx = \frac{2}{3} L$$

$$\therefore \vec{r}_{CM} = \left(\frac{2}{3}, 0\right) \times L$$

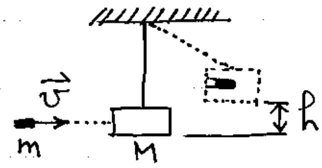
羅

Wolfson Example 9.9

Ballistic pendulum (衝擊擺)

用於測量子彈 m 的 speed, 如右圖.

則 v=?



$\Rightarrow$ 1+1=1 : 動量守恒

0 → h : mechanical energy of the system 守恒

設 m 在進入 M 的瞬間, 兩者一体的 speed = V,

則 $mv = (m+M)V$, $\therefore V = \frac{m}{m+M} v$ (動量守恒)

$$\Delta K + \Delta U = 0 = (K_f - K_i) + (U_f - U_i) = \left[0 - \frac{1}{2}(m+M)V^2\right] + \left[(m+M)gh - 0\right]$$

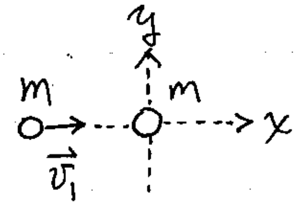
$$\therefore V^2 = 2gh = \left(\frac{m}{m+M}\right)^2 \cdot v^2 \Rightarrow v = \left(\frac{m+M}{m}\right)\sqrt{2gh}.$$

羅

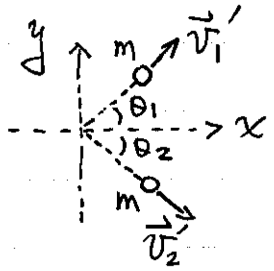
Wolfson Example 9.11

2D 等質量碰撞後的夾角 = 90°

Before



After



show that $\theta_1+\theta_2=\frac{\pi}{2}$

Before: $\vec{p}_1=m\vec{v}_1$

After: $\vec{p}_1'+\vec{p}_2'=m\vec{v}_1'+m\vec{v}_2'$

$\vec{p}$ 守恒: $\vec{p}_1=\vec{p}_1'+\vec{p}_2'$ —(a)

k 守恒: $k_1=k_1'+k_2' \rightarrow \frac{p_1^2}{2m}=\frac{p_1'^2}{2m}+\frac{p_2'^2}{2m}$, $\therefore p_1^2=p_1'^2+p_2'^2$ —(b)

from $(a)^2$ $\quad p_1^2=\vec{p}_1\cdot\vec{p}_1=p_1'^2+p_2'^2+2\vec{p}_1'\cdot\vec{p}_2'=p_1'^2+p_2'^2$

$\therefore \vec{p}_1'\cdot\vec{p}_2'=0 \Rightarrow \vec{v}_1'\perp\vec{v}_2'$, $\therefore \theta_1+\theta_2=90°$.

羅